



CAUCHY-KOWALEVSKI EXTENSIONS, FUETER'S  
THEOREMS AND BOUNDARY VALUES OF SPECIAL  
SYSTEMS IN CLIFFORD ANALYSIS

by

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*A mis padres, mi esposa Barbara,  
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# Introduction

In the year 1878 William Kingdon Clifford (1845-1879) introduced the algebras named after him which may be regarded as a generalization of the complex numbers and Hamilton's quaternions (see [33]). They are a type of finite-dimensional associative algebra and have important applications in a variety of fields including geometry and theoretical physics.

Clifford analysis is a successful generalization to higher dimension of the theory of holomorphic functions in the complex plane. It involves the study of functions on Euclidean space with values in a Clifford algebra. For a thorough treatment of this function theory we refer the reader to e.g. [26, 34, 45, 61, 64, 65].

The main objects of study in Clifford analysis are the so-called monogenic functions which may be described as null solutions of the Dirac operator, the latter being the higher dimensional analogue of the Cauchy-Riemann operator.

Some of the earlier results on Clifford analysis were obtained by Dixon [37], Moisil and Théodoresco [83], Fueter [55, 56], Iftimie [71], Hestenes [69] and Delanghe [39, 40, 41, 42]. The basic theory of Clifford analysis was developed in the book by Brackx, Delanghe and Sommen [26] in 1982. This is the first book written on Clifford analysis and it is the basic reference work on the subject. Nowadays Clifford analysis is a well established mathematical discipline as well as an active area of scientific research.

The subject of this thesis fits in the framework of Clifford analysis. The first part deals with some techniques to generate monogenic functions and

the second part is devoted to the study of extension theorems for special systems arising in Clifford analysis.

In order to make the reader familiar with the concepts used in this thesis, the first chapter contains a review of the definitions and fundamental results concerning Clifford algebras, Clifford analysis, the Cauchy type integral and the singular integral operator.

Despite the fact that Clifford analysis generalizes the most important features of classical complex analysis, monogenic functions do not enjoy all properties of holomorphic functions of one complex variable. For instance, due to the non-commutativity of the Clifford algebras, the product of two monogenic functions is in general not monogenic. It is therefore natural to look for specific techniques to construct monogenic functions.

There are several techniques available to generate monogenic functions, see [26, 43, 45]. Two of those techniques are considered in this thesis: the Cauchy-Kowalevski extension problem and Fueter's theorem. We also introduce a new technique leading to so-called steering monogenic functions.

The first technique mentioned consists in monogenically extending analytic functions defined on a given subset in  $\mathbb{R}^{m+1}$  of positive codimension. The second one gives a method to generate monogenic functions starting from a holomorphic function in the upper half of the complex plane. Finally, steering monogenic functions can be roughly described as a class of monogenic functions generated from families of complex-valued functions which are closed under conjugation and under the action of the Cauchy-Riemann operator.

In Chapter 2 we introduce the notion of steering monogenic functions and we discuss the Cauchy-Kowalevski extension around special surfaces of codimension two.

In Chapter 3 we provide an alternative proof for Fueter's theorem. Using the main idea of this proof, we also establish a new generalization of Fueter's theorem. Some examples of applications are also computed including a closed formula for the Cauchy-Kowalevski extension of the Gauss-distribution in  $\mathbb{R}^m$ .

Chapter 4 deals with a recent refinement of the theory of monogenic functions: Hermitean Clifford analysis. It studies so-called Hermitean monogenic functions which are simultaneous null solutions of two mutually related Euclidean Dirac operators (see [24, 25, 27, 101, 102]). We derive two criteria providing necessary and sufficient conditions for the existence of a Hermitean monogenic extension of a continuous function defined on a surface in  $\mathbb{R}^m$ ,  $m = 2n$ . These characterizations are then used to study the jump problem in this context.

In the even dimensional case the Dirac equation may be reduced to the so-called isotonic Dirac system in which different Dirac operators in half the dimension act from both sides on the unknown function. Solutions of this system are called isotonic functions and are closely related with Hermitean monogenic functions. Chapter 5 is devoted to the study of these functions. First, we obtain an integral representation formula. Next, some direct applications of this formula are indicated. The remainder of this chapter is devoted to the study of the isotonic Cauchy type integral and its singular version.

Finally, in the last chapter, extension theorems for holomorphic and bi-regular functions are studied. The latter may be considered as monogenic functions of two higher dimensional variables. As holomorphic and biregular functions are particular cases of isotonic functions, the results obtained in Chapter 5 enable us to get simplified and elegant proofs.



# Chapter 1

## Some basic elements of Clifford analysis

This chapter contains a summary of the Clifford analysis theory we will use. For a thorough treatment we refer the reader to [26, 34, 45, 61, 64, 65].

### 1.1 Clifford algebras

Clifford algebras, also called geometric algebras, extend the real number system to include vectors and their products. Clifford algebras have important applications in geometry and theoretical physics. They are named after the English geometer and philosopher W. K. Clifford (see [33]).

We denote by  $\mathbb{R}_{0,m}$  ( $m \in \mathbb{N}$ ) the real Clifford algebra constructed over the orthonormal basis  $(e_1, \dots, e_m)$  of the Euclidean space  $\mathbb{R}^m$ . The basic axiom of this associative but non-commutative algebra is that the product of a vector with itself equals its squared length up to a minus sign, i.e. for any vector  $\underline{x} = \sum_{j=1}^m x_j e_j$  in  $\mathbb{R}^m$ , we have that

$$\underline{x}^2 = -|\underline{x}|^2 = -\sum_{j=1}^m x_j^2.$$

It thus follows that the elements of the basis submit to the multiplication rules

$$\begin{aligned} e_j^2 &= -1, & j &= 1, \dots, m, \\ e_j e_k + e_k e_j &= 0, & 1 \leq j \neq k \leq m. \end{aligned}$$

A basis for the algebra is then given by the elements

$$e_A = e_{j_1} \cdots e_{j_k},$$

where  $A = \{j_1, \dots, j_k\} \subset \{1, \dots, m\}$  is such that  $j_1 < \dots < j_k$ . For the empty set  $\emptyset$ , we put  $e_\emptyset = e_0 = 1$ , the latter being the identity element. It follows that the dimension of  $\mathbb{R}_{0,m}$  is  $2^m$ .

Any Clifford number  $a \in \mathbb{R}_{0,m}$  may thus be written as

$$a = \sum_A a_A e_A, \quad a_A \in \mathbb{R}.$$

For each  $k \in \{0, 1, \dots, m\}$ , we call

$$\mathbb{R}_{0,m}^{(k)} = \left\{ a \in \mathbb{R}_{0,m} : a = \sum_{|A|=k} a_A e_A \right\}$$

the subspace of  $k$ -vectors, i.e. the space spanned by the products of  $k$  different basis vectors. In particular, the 0-vectors and 1-vectors are simply called scalars and vectors respectively.

An important subspace of the real Clifford algebra  $\mathbb{R}_{0,m}$  is the so-called space of paravectors  $\mathbb{R} \oplus \mathbb{R}_{0,m}^{(1)}$ , being sums of scalars and vectors. Observe that  $\mathbb{R}^{m+1}$  may be naturally identified with  $\mathbb{R} \oplus \mathbb{R}_{0,m}^{(1)}$  by associating to any element  $(x_0, \underline{x}) = (x_0, x_1, \dots, x_m) \in \mathbb{R}^{m+1}$  the paravector  $x = x_0 + \underline{x}$ .

Note that

$$\mathbb{R}_{0,m} = \bigoplus_{k=0}^m \mathbb{R}_{0,m}^{(k)}$$

and hence for any  $a \in \mathbb{R}_{0,m}$

$$a = \sum_{k=0}^m [a]_k,$$

where  $[a]_k$  is the projection of  $a$  on  $\mathbb{R}_{0,m}^{(k)}$ .

The product of two Clifford vectors  $\underline{x} = \sum_{j=1}^m x_j e_j$  and  $\underline{y} = \sum_{j=1}^m y_j e_j$  splits into a scalar part and a 2-vector or so-called bivector part

$$\underline{x} \underline{y} = \underline{x} \bullet \underline{y} + \underline{x} \wedge \underline{y}, \quad (1.1)$$

where

$$\underline{x} \bullet \underline{y} = -\langle \underline{x}, \underline{y} \rangle = -\sum_{j=1}^m x_j y_j$$

equals, up to a minus sign, the standard Euclidean inner product between  $\underline{x}$  and  $\underline{y}$ , while

$$\underline{x} \wedge \underline{y} = \sum_{j=1}^m \sum_{k=j+1}^m e_j e_k (x_j y_k - x_k y_j)$$

represents the standard outer (or wedge) product between them.

More generally, for a vector  $\underline{x}$  and a  $k$ -vector  $Y_k$ , the inner and outer product between  $\underline{x}$  and  $Y_k$  are defined by

$$\underline{x} \bullet Y_k = \begin{cases} [\underline{x} Y_k]_{k-1} & \text{for } k \geq 1 \\ 0 & \text{for } k = 0 \end{cases} \quad \text{and} \quad \underline{x} \wedge Y_k = [\underline{x} Y_k]_{k+1}.$$

In a similar way,

$$Y_k \bullet \underline{x} = \begin{cases} [Y_k \underline{x}]_{k-1} & \text{for } k \geq 1 \\ 0 & \text{for } k = 0 \end{cases} \quad \text{and} \quad Y_k \wedge \underline{x} = [Y_k \underline{x}]_{k+1}.$$

We thus have that

$$\begin{aligned} \underline{x} Y_k &= \underline{x} \bullet Y_k + \underline{x} \wedge Y_k, \\ Y_k \underline{x} &= Y_k \bullet \underline{x} + Y_k \wedge \underline{x}, \end{aligned}$$

where also

$$\begin{aligned}\underline{x} \bullet Y_k &= (-1)^{k-1} Y_k \bullet \underline{x}, \\ \underline{x} \wedge Y_k &= (-1)^k Y_k \wedge \underline{x}.\end{aligned}$$

Two important examples of real Clifford algebras are the field of complex numbers  $\mathbb{C}$  and the skew field of quaternions  $\mathbb{H}$ . Indeed, note that  $\mathbb{R}_{0,1}$  is a two-dimensional algebra generated by a single vector  $e_1$  which squares to  $-1$ , and therefore  $\mathbb{R}_{0,1}$  is isomorphic to  $\mathbb{C}$ . On the other hand,  $\mathbb{R}_{0,2}$  is a four-dimensional algebra spanned by  $\{1, e_1, e_2, e_1 e_2\}$ . The latter three elements square to  $-1$  and all anticommute, and so the algebra  $\mathbb{R}_{0,2}$  is isomorphic to the quaternions  $\mathbb{H}$ .

Three (anti)-involutions are defined on  $\mathbb{R}_{0,m}$ : the main involution, the reversion and the conjugation.

The main involution  $a \rightarrow \tilde{a}$  is given by

$$\tilde{a} = \sum_A a_A \tilde{e}_A,$$

where  $\tilde{e}_A = (-1)^k e_A$  if  $|A| = k$ .

The reversion  $a \rightarrow a^*$  is given by

$$a^* = \sum_A a_A e_A^*,$$

where  $e_A^* = e_{j_k} \cdots e_{j_1} = (-1)^{\frac{(k-1)k}{2}} e_A$  if  $e_A = e_{j_1} \cdots e_{j_k}$ .

Finally, the conjugation  $a \rightarrow \bar{a}$  is a combination of the main involution and the reversion introduced above. It is defined as

$$\bar{a} = (\tilde{a})^* = \sum_A a_A (\tilde{e}_A)^*.$$

One easily checks that

$$\widetilde{ab} = \tilde{a}\tilde{b},$$



$$\begin{aligned}(ab)^* &= b^* a^*, \\ \overline{ab} &= \bar{b}\bar{a},\end{aligned}$$

for any  $a, b \in \mathbb{R}_{0,m}$ .

By means of the conjugation, a norm  $|a|$  may be defined for each  $a \in \mathbb{R}_{0,m}$  by putting

$$|a|^2 = [a\bar{a}]_0 = \sum_A a_A^2.$$

It immediately follows that for any  $a, b \in \mathbb{R}_{0,m}$

$$|a + b| \leq |a| + |b| \quad \text{and} \quad |ab| \leq 2^{\frac{m}{2}} |a| |b|.$$

In this thesis we also deal with the complex Clifford algebra  $\mathbb{C}_m$ , which may be defined as

$$\mathbb{C}_m = \mathbb{C} \otimes \mathbb{R}_{0,m} = \mathbb{R}_{0,m} \oplus i \mathbb{R}_{0,m}.$$

Any complex Clifford number  $a \in \mathbb{C}_m$  may thus be represented as

$$a = \sum_A a_A e_A, \quad a_A \in \mathbb{C}.$$

All concepts introduced above in the context of  $\mathbb{R}_{0,m}$  may be reformulated in the case of  $\mathbb{C}_m$  in a very similar way. The major difference lies in the conjugation, where the additional rule  $\bar{\bar{i}} = -i$  has to be included.

It is worth pointing out that for  $m \geq 3$  the real Clifford algebra  $\mathbb{R}_{0,m}$  has zero divisors. Indeed, it is easily seen that  $e_{123}$  squares to 1 and hence

$$(1 + e_{123})(1 - e_{123}) = (1 - e_{123})(1 + e_{123}) = 0.$$

Thus for  $m \geq 3$  not every Clifford number in  $\mathbb{R}_{0,m}$  has a multiplicative inverse. Fortunately, any non-zero paravector  $x$  does have a multiplicative inverse given by

$$x^{-1} = \frac{\bar{x}}{|x|^2}.$$

In the case of  $\mathbb{C}_m$  we also have that

$$(1 + i\underline{\omega})(1 - i\underline{\omega}) = (1 - i\underline{\omega})(1 + i\underline{\omega}) = 0,$$

with  $\underline{\omega} = \frac{x}{|x|}$ .

## 1.2 Monogenic functions

Monogenic functions are the central object of study in Clifford analysis. The concept of monogenicity of a function may be seen as the higher dimensional counterpart of holomorphy in the complex plane.

The functions under consideration are defined on an open subset of  $\mathbb{R}^m$  or  $\mathbb{R}^{m+1}$  and take values in the Clifford algebra  $\mathbb{R}_{0,m}$  or in its complexification  $\mathbb{C}_m$ . They are of the form

$$f = \sum_A f_A e_A,$$

where the functions  $f_A$  are  $\mathbb{R}$ -valued or  $\mathbb{C}$ -valued.

Whenever a property such as continuity, differentiability, etc. is ascribed to  $f$  it is clear that in fact all the components  $f_A$  possess the cited property.

Next, we introduce the Dirac operator

$$\partial_{\underline{x}} = \sum_{j=1}^m e_j \partial_{x_j}$$

and the generalized Cauchy-Riemann operator

$$\partial_x = \partial_{x_0} + \partial_{\underline{x}}.$$

These operators factorize the Laplace operator in the sense that

$$\Delta_{\underline{x}} = \sum_{j=1}^m \partial_{x_j}^2 = -\partial_{\underline{x}}^2 \quad (1.2)$$

and

$$\Delta_x = \partial_{x_0}^2 + \Delta_{\underline{x}} = \partial_x \bar{\partial}_x = \bar{\partial}_x \partial_x. \quad (1.3)$$

**Definition 1.1** *A function  $f(\underline{x})$  (resp.  $f(x)$ ) defined and continuously differentiable in an open set  $\Omega$  of  $\mathbb{R}^m$  (resp.  $\mathbb{R}^{m+1}$ ) and taking values in  $\mathbb{R}_{0,m}$*

or  $\mathbb{C}_m$ , is called a *left monogenic function* in  $\Omega$  if and only if it fulfills in  $\Omega$  the equation

$$\partial_{\underline{x}} f \equiv \sum_{j=1}^m \sum_A e_j e_A \partial_{x_j} f_A = 0 \quad (\text{resp. } \partial_x f \equiv \sum_{j=0}^m \sum_A e_j e_A \partial_{x_j} f_A = 0).$$

Note that in view of the non-commutativity of  $\mathbb{R}_{0,m}$  and  $\mathbb{C}_m$  a notion of right monogenicity may be defined in a similar way by letting act the Dirac operator or the generalized Cauchy-Riemann operator from the right.

Nevertheless, we will just say “ $f$  is monogenic in  $\Omega$ ” instead of “ $f$  is left monogenic in  $\Omega$ ”.

From (1.2) and (1.3) it follows that any monogenic function in  $\Omega$  is harmonic in  $\Omega$  and hence real-analytic in  $\Omega$ .

To fix the ideas let us examine two special cases of monogenic functions. First, if  $m = 1$ , then a function  $f : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}_{0,1}$  is of the form  $f = f_0 + f_1 e_1$  and  $\partial_x = \partial_{x_0} + e_1 \partial_{x_1}$ , so the monogenicity of  $f$  reduces to the system

$$\begin{cases} \partial_{x_0} f_0 - \partial_{x_1} f_1 = 0 \\ \partial_{x_1} f_0 + \partial_{x_0} f_1 = 0 \end{cases}$$

which is nothing else but the classical Cauchy-Riemann system for holomorphic functions of one complex variable. Next, let  $f$  be a vector-valued function in  $\Omega \subset \mathbb{R}^m$ , i.e.

$$f(\underline{x}) = \sum_{j=1}^m f_j(\underline{x}) e_j.$$

Then, from (1.1) we obtain

$$\partial_{\underline{x}} f = \partial_{\underline{x}} \bullet f + \partial_{\underline{x}} \wedge f.$$

Claiming that  $\partial_{\underline{x}} f = 0$  in  $\Omega$  is thus equivalent to saying that its components  $f_j$ ,  $j = 1, \dots, m$ , satisfy the so-called Riesz system

$$\begin{cases} \sum_{j=1}^m \partial_{x_j} f_j = 0, \\ \partial_{x_j} f_k - \partial_{x_k} f_j = 0, \quad 1 \leq j \neq k \leq m. \end{cases}$$

It is clear that the set of  $\mathbb{R}_{0,m}$ -valued (resp.  $\mathbb{C}_m$ -valued) monogenic functions in  $\Omega$  provided with the classical rules for addition and for multiplication with Clifford scalars is a right  $\mathbb{R}_{0,m}$ -module (resp.  $\mathbb{C}_m$ -module).

We emphasize that the product of two monogenic functions is, in general, not monogenic.

For a vector-valued differentiable function  $\underline{f} = \sum_{j=1}^m f_j e_j$  and a Clifford algebra-valued differentiable function  $g$ , we have the following Leibniz rule (the general version will be given in the third chapter)

$$\partial_{\underline{x}}(\underline{f}g) = (\partial_{\underline{x}}\underline{f})g - \underline{f}(\partial_{\underline{x}}g) - 2 \sum_{j=1}^m f_j (\partial_{x_j}g). \quad (1.4)$$

Indeed,

$$\partial_{\underline{x}}(\underline{f}g) = \sum_{j=1}^m e_j ((\partial_{x_j}\underline{f})g + \underline{f}(\partial_{x_j}g)) = (\partial_{\underline{x}}\underline{f})g + \sum_{j=1}^m e_j \underline{f}(\partial_{x_j}g),$$

which results in (1.4) on account of the equality

$$e_j \underline{f} = -\underline{f}e_j - 2f_j, \quad j = 1, \dots, m.$$

In particular, for  $\underline{f} = \underline{x}$  we have

$$\partial_{\underline{x}}(\underline{x}g) = -mg - \underline{x}(\partial_{\underline{x}}g) - 2E_{\underline{x}}g, \quad (1.5)$$

$E_{\underline{x}} = \sum_{j=1}^m x_j \partial_{x_j}$  being the Euler operator.

Using (1.4) we may also prove the following simple but interesting statement: if  $f$  is a monogenic function in some open connected set  $\Omega$  of  $\mathbb{R}^m$  such that  $e_j f$  is also monogenic in  $\Omega$  for each  $j = 1, \dots, m$ , then  $f$  is a constant in  $\Omega$ . Indeed,  $e_j f$  being monogenic, we have, for each  $j = 1, \dots, m$

$$0 = \partial_{\underline{x}}(e_j f(\underline{x})) = -2\partial_{x_j}f(\underline{x}), \quad \underline{x} \in \Omega.$$

From the above it follows that all first order partial derivatives of  $f$  vanish, and consequently  $f$  is a constant function in  $\Omega$ .

Let  $\Gamma_{\underline{x}}$  denote the spherical Dirac operator (or Gamma operator) on the unit sphere  $S^{m-1}$  in  $\mathbb{R}^m$ , i.e.

$$\Gamma_{\underline{x}} = -\underline{x} \wedge \partial_{\underline{x}} = -\sum_{j=1}^m \sum_{k=j+1}^m e_j e_k (x_j \partial_{x_k} - x_k \partial_{x_j}).$$

From (1.1) we see that

$$\underline{x} \partial_{\underline{x}} = -E_{\underline{x}} - \Gamma_{\underline{x}}. \quad (1.6)$$

Introducing spherical coordinates  $\underline{x} = r\underline{\omega}$  ( $r = |\underline{x}|$ ,  $\underline{\omega} \in S^{m-1}$ ) and using the fact that  $E_{\underline{x}} = r\partial_r$ , we obtain the spherical decomposition of the Dirac operator

$$\partial_{\underline{x}} = \underline{\omega} \left( \partial_r + \frac{1}{r} \Gamma_{\underline{x}} \right). \quad (1.7)$$

Next, we recall two properties of the spherical Dirac operator  $\Gamma_{\underline{x}}$  that are frequently used in calculations (see [45]). If  $f(r)$  is a function of  $r$ , then

$$\begin{aligned} \Gamma_{\underline{x}} f(r) &= -\sum_{j=1}^m \sum_{k=j+1}^m e_j e_k (x_j \partial_{x_k} f(r) - x_k \partial_{x_j} f(r)) \\ &= -\sum_{j=1}^m \sum_{k=j+1}^m e_j e_k (x_j (\partial_r f(r)) (\partial_{x_k} r) - x_k (\partial_r f(r)) (\partial_{x_j} r)) \\ &= -\sum_{j=1}^m \sum_{k=j+1}^m e_j e_k \left( x_j (\partial_r f(r)) \frac{x_k}{r} - x_k (\partial_r f(r)) \frac{x_j}{r} \right). \end{aligned}$$

So that

$$\Gamma_{\underline{x}} f(r) = 0. \quad (1.8)$$

On account of the above remark and using (1.7), we see that

$$\partial_{\underline{x}} f(r) = \underline{\omega} \partial_r f(r).$$

From (1.6) and applying (1.5) we also get

$$\begin{aligned}
\Gamma_{\underline{x}}(\underline{\omega}f(\underline{x})) &= \Gamma_{\underline{x}}\left(\frac{1}{r}\right)\underline{x}f(\underline{x}) + \frac{1}{r}\Gamma_{\underline{x}}(\underline{x}f(\underline{x})) \\
&= -\frac{1}{r}\left(\underline{x}\partial_{\underline{x}}(\underline{x}f(\underline{x})) + \mathbf{E}_{\underline{x}}(\underline{x}f(\underline{x}))\right) \\
&= \frac{1}{r}\left(m\underline{x}f(\underline{x}) + \underline{x}^2(\partial_{\underline{x}}f(\underline{x})) + 2\underline{x}\mathbf{E}_{\underline{x}}f(\underline{x})\right. \\
&\quad \left.- (\mathbf{E}_{\underline{x}}\underline{x})f(\underline{x}) - \underline{x}\mathbf{E}_{\underline{x}}f(\underline{x})\right) \\
&= \underline{\omega}\left((m-1)f(\underline{x}) + \underline{x}(\partial_{\underline{x}}f(\underline{x})) + \mathbf{E}_{\underline{x}}f(\underline{x})\right).
\end{aligned}$$

This gives

$$\Gamma_{\underline{x}}(\underline{\omega}f(\underline{x})) = (m-1)\underline{\omega}f(\underline{x}) - \underline{\omega}\Gamma_{\underline{x}}f(\underline{x}). \quad (1.9)$$

By the above and using (1.7), we can assert that

$$\partial_{\underline{x}}\underline{\omega} = \frac{\underline{\omega}}{r}\Gamma_{\underline{x}}\underline{\omega} = (m-1)\frac{\underline{\omega}^2}{r} = -\frac{(m-1)}{r}.$$

Let us now consider monogenic functions of the form

$$(A(x_0, r) + \underline{\omega}B(x_0, r))P_k(\underline{x}), \quad (1.10)$$

where  $A(x_0, r)$  and  $B(x_0, r)$  are  $\mathbb{R}$ -valued continuously differentiable functions, and  $P_k(\underline{x})$  is a homogeneous monogenic polynomial of degree  $k$  in  $\mathbb{R}^m$ , i.e.

$$\begin{aligned}
\partial_{\underline{x}}P_k(\underline{x}) &= 0, & \underline{x} &\in \mathbb{R}^m, \\
P_k(t\underline{x}) &= t^k P_k(\underline{x}), & t &\in \mathbb{R}.
\end{aligned}$$

Functions (1.10) are called axial monogenic functions of degree  $k$  (see [77, 111, 116]) and they generate monogenic functions in axially symmetric domains.

Note that

$$\begin{aligned}
\partial_{\underline{x}}[(A + \underline{\omega}B)P_k(\underline{x})] &= (\partial_{\underline{x}}A)P_k(\underline{x}) + A(\partial_{\underline{x}}P_k(\underline{x})) \\
&\quad + (\partial_{\underline{x}}B)\underline{\omega}P_k(\underline{x}) + B(\partial_{\underline{x}}(\underline{\omega}P_k(\underline{x}))) \\
&= \underline{\omega}(\partial_r A)P_k(\underline{x}) - (\partial_r B)P_k(\underline{x}) + B(\partial_{\underline{x}}(\underline{\omega}P_k(\underline{x}))),
\end{aligned}$$

where also

$$\begin{aligned}\partial_{\underline{x}}(\underline{\omega}P_k(\underline{x})) &= (\partial_{\underline{x}}\underline{\omega})P_k(\underline{x}) - \underline{\omega}(\partial_{\underline{x}}P_k(\underline{x})) - \frac{2}{r}\mathbf{E}_{\underline{x}}P_k(\underline{x}) \\ &= -\frac{2k+m-1}{r}P_k(\underline{x}),\end{aligned}$$

which follows from (1.4) and Euler's homogeneous function theorem.

We thus get

$$\partial_{\underline{x}}[(A + \underline{\omega}B)P_k(\underline{x})] = \left[ \underline{\omega}\partial_r A - \left( \partial_r B + \frac{2k+m-1}{r}B \right) \right] P_k(\underline{x}).$$

For this reason

$$\begin{aligned}\partial_x[(A + \underline{\omega}B)P_k(\underline{x})] \\ = \left[ \left( \partial_{x_0}A - \partial_r B - \frac{2k+m-1}{r}B \right) + \underline{\omega}(\partial_{x_0}B + \partial_r A) \right] P_k(\underline{x})\end{aligned}$$

and so the assumed monogenicity requires the functions  $A$  and  $B$  to satisfy the Vekua-type system

$$\begin{cases} \partial_{x_0}A - \partial_r B &= \frac{2k+m-1}{r}B \\ \partial_{x_0}B + \partial_r A &= 0. \end{cases} \quad (1.11)$$

We will also deal with another technique to generate monogenic functions in  $\mathbb{R}^{m+1}$ : the so-called Cauchy-Kowalevski extension (CK-extension) problem.

The CK-extension problem consists in finding a monogenic extension  $g^*$  of an analytic function  $g$  defined on a given subset in  $\mathbb{R}^{m+1}$  of positive codimension (see e.g. [26, 35, 44, 45, 72, 110, 112, 118]).

For analytic functions  $g$  on the plane  $\{(x_0, \underline{x}) \in \mathbb{R}^{m+1} : x_0 = 0\}$  the above problem may be stated as follows: find  $g^*$  such that

$$\begin{aligned}\partial_{x_0}g^* &= -\partial_{\underline{x}}g^*, \quad \text{in } \mathbb{R}^{m+1} \\ g^*(0, \underline{x}) &= g(\underline{x}).\end{aligned}$$

Formally solving this equation we obtain

$$\begin{aligned} g^*(x_0, \underline{x}) &= \exp(-x_0 \partial_{\underline{x}}) g(\underline{x}) \\ &= \sum_{k=0}^{\infty} \frac{(-x_0)^k}{k!} \partial_{\underline{x}}^k g(\underline{x}). \end{aligned} \quad (1.12)$$

It may be proved that (1.12) is a monogenic extension of the function  $g$  in  $\mathbb{R}^{m+1}$  (see [26]). Moreover, by the uniqueness theorem for monogenic functions this extension is also unique.

More in general, the CK-extension for analytic functions on an analytic  $m$ -surface in  $\mathbb{R}^{m+1}$  exists and it is also unique (see [112]). For the case of surfaces with higher codimension this problem has not yet been solved, except in the flat case (see [44, 45]).

Before introducing the basic integral formulae of Clifford analysis, we need a few definitions from geometric measure theory. Geometric measure theory can be roughly described as differential geometry, generalized through measure theory to deal with maps and surfaces that are not necessarily smooth, and applied to the calculus of variations. For a detailed exposition we refer the reader to [46, 47, 52, 53, 54, 62, 67, 81, 84, 109].

Let  $A$  be a subset of  $\mathbb{R}^m$  and let  $k \leq m$  be a positive integer. With each  $\delta > 0$  we associate the infimum of all numbers of the form

$$\sum_{j=1}^{\infty} 2^{-k} \alpha(k) (\text{diam}(B_j))^k,$$

where

$$A \subset \bigcup_{j=1}^{\infty} B_j, \quad \text{with } \text{diam}(B_j) < \delta, \quad j = 1, 2, 3, \dots$$

Here  $\alpha(k)$  denotes the volume of the unit sphere in  $\mathbb{R}^k$ , and for any non-empty subset  $B$  of  $\mathbb{R}^m$ , its diameter is defined as

$$\text{diam}(B) = \sup\{|\underline{x} - \underline{y}| : \underline{x}, \underline{y} \in B\}.$$



If  $\delta$  tends to zero, this infimum is non-decreasing; it approaches the limit  $\mathcal{H}^k(A)$ , which is by definition the  $k$ -dimensional Hausdorff measure of  $A$ .

In 1918, F. Hausdorff introduced this  $k$ -dimensional measure on  $\mathbb{R}^m$ , defined for all subsets, and coinciding for “nice” subsets, with the usual  $k$ -dimensional surface area. When  $k = m$ , it equals the Lebesgue measure.

The definition of Hausdorff measure extends to any non-negative real number  $k$ , taking

$$\alpha(k) = \frac{\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2} + 1)},$$

where  $\Gamma$  stands for the usual Gamma function. Observe that  $\mathcal{H}^0$  equals the counting measure, i.e.  $\mathcal{H}^0(A)$  is the number of elements of  $A$ .

Throughout the thesis we assume  $\Omega^+$  to be a simply connected bounded and open set in  $\mathbb{R}^m$ ,  $\Omega^- = \mathbb{R}^m \setminus \overline{\Omega^+}$ ,  $\Sigma$  is the boundary surface of  $\Omega^+$ , and  $\mathcal{H}^{m-1}(\Sigma) < \infty$ .

The open ball of radius  $\delta > 0$  centered at a point  $\underline{x}$  in  $\mathbb{R}^m$  will be denoted by  $B(\underline{x}, \delta)$  and is defined by

$$B(\underline{x}, \delta) = \{\underline{y} \in \mathbb{R}^m : |\underline{y} - \underline{x}| < \delta\}.$$

**Definition 1.2** *A unit vector  $\underline{w}$  is said to be an exterior normal of  $\Omega^+$  at  $\underline{x} \in \Sigma$  (in the sense of Federer) if and only if*

$$\delta^{-m} \mathcal{L}^m(\{\underline{y} : \langle \underline{y} - \underline{x}, \underline{w} \rangle < 0, \underline{y} \in B(\underline{x}, \delta) \setminus \Omega^+\}) \rightarrow 0$$

and

$$\delta^{-m} \mathcal{L}^m(\{\underline{y} : \langle \underline{y} - \underline{x}, \underline{w} \rangle > 0, \underline{y} \in B(\underline{x}, \delta) \cap \Omega^+\}) \rightarrow 0$$

as  $\delta \rightarrow 0+$ . Here  $\mathcal{L}^m$  denotes the  $m$ -dimensional Lebesgue measure over  $\mathbb{R}^m$ .

Such a unit vector  $\underline{w}$ , if it exists, is uniquely determined by  $\Omega^+$  and  $\underline{x}$ , and will be denoted by  $\underline{\nu}(\underline{x})$ . In case no such  $\underline{w}$  exists,  $\underline{\nu}(\underline{x})$  is the

null vector. This defines for each  $\underline{x} \in \Sigma$  a vector  $\underline{\nu}(\underline{x})$  with components  $\nu_1(\underline{x}), \dots, \nu_m(\underline{x})$ , i.e.

$$\underline{\nu}(\underline{x}) = \sum_{j=1}^m \nu_j(\underline{x}) e_j.$$

We note that if  $\underline{x}$  is a smooth boundary point of  $\Sigma$ , then  $\underline{\nu}(\underline{x})$  is the usual exterior normal.

In order to work with sets with very general boundaries the following version of the Gauss-Green Theorem provided by H. Federer will be needed (see [48, 49, 50, 51]). For other generalizations we refer the reader to e.g. [38, 68, 93, 94, 120].

**Theorem 1.1 (Gauss-Green Theorem)** *If the vector-valued function  $\underline{F}$  is differentiable in  $\Omega^+$ , continuous on  $\overline{\Omega}^+$ , and such that*

$$\int_{\Omega^+} |\operatorname{div} \underline{F}(\underline{x})| d\mathcal{L}^m(\underline{x}) < \infty,$$

*then*

$$\int_{\Omega^+} \operatorname{div} \underline{F}(\underline{x}) d\mathcal{L}^m(\underline{x}) = \int_{\Sigma} \langle \underline{F}(\underline{x}), \underline{\nu}(\underline{x}) \rangle d\mathcal{H}^{m-1}(\underline{x}).$$

We are now ready to introduce the basic integral formulae of Clifford analysis. But first we recall that the fundamental solution of the Dirac operator  $\partial_{\underline{x}}$  is the  $L_1^{\operatorname{loc}}$ -function

$$E(\underline{x}) = -\frac{1}{\omega_m} \frac{\underline{x}}{|\underline{x}|^m}, \quad \underline{x} \in \mathbb{R}^m \setminus \{0\},$$

where  $\omega_m$  is the area of the unit sphere  $S^{m-1}$  in  $\mathbb{R}^m$ . It is easily seen that  $E(\underline{x})$  is monogenic in  $\mathbb{R}^m \setminus \{0\}$  and vanishes at infinity.

**Theorem 1.2 (Clifford-Gauss-Green Theorem)** *Let  $f$  and  $g$  be continuously differentiable functions in  $\Omega^+$ , which are continuous on  $\overline{\Omega}^+$ , and moreover satisfy*

$$\int_{\Omega^+} |(f(\underline{x}) \partial_{\underline{x}})g(\underline{x}) + f(\underline{x})(\partial_{\underline{x}}g(\underline{x}))| d\mathcal{L}^m(\underline{x}) < \infty,$$

then

$$\int_{\Sigma} f(\underline{x}) \underline{\nu}(\underline{x}) g(\underline{x}) d\mathcal{H}^{m-1}(\underline{x}) = \int_{\Omega^+} [(f(\underline{x}) \partial_{\underline{x}}) g(\underline{x}) + f(\underline{x}) (\partial_{\underline{x}} g(\underline{x}))] d\mathcal{L}^m(\underline{x}).$$

**Theorem 1.3 (Borel-Pompeiu Formula)** *Let  $f$  be a continuously differentiable function in  $\Omega^+$ , continuous on  $\overline{\Omega^+}$ , and such that*

$$\int_{\Omega^+} |\partial_{\underline{x}} f(\underline{x})| d\mathcal{L}^m(\underline{x}) < \infty.$$

Then

$$\begin{aligned} \int_{\Sigma} E(\underline{y} - \underline{x}) \underline{\nu}(\underline{y}) f(\underline{y}) d\mathcal{H}^{m-1}(\underline{y}) - \int_{\Omega^+} E(\underline{y} - \underline{x}) \partial_{\underline{y}} f(\underline{y}) d\mathcal{L}^m(\underline{y}) \\ = \begin{cases} f(\underline{x}) & \text{for } \underline{x} \in \Omega^+, \\ 0 & \text{for } \underline{x} \in \Omega^-. \end{cases} \end{aligned}$$

**Theorem 1.4 (Cauchy's Integral Formula)** *Suppose that  $f$  is a continuous function on  $\overline{\Omega^+}$ . If  $f$  is monogenic in  $\Omega^+$ , then*

$$f(\underline{x}) = \int_{\Sigma} E(\underline{y} - \underline{x}) \underline{\nu}(\underline{y}) f(\underline{y}) d\mathcal{H}^{m-1}(\underline{y}), \quad \underline{x} \in \Omega^+.$$

As in classical complex analysis, Cauchy's Integral Formula is an essential tool in Clifford analysis. Applications of this result include the Mean Value Theorem, Liouville's Theorem, the Maximum Modulus Theorem and Weierstrass' Convergence Theorem (see [26]).

In a similar way the integral formulae for the generalized Cauchy-Riemann operator  $\partial_x$  may be introduced.

### 1.3 The Cauchy type integral

One of the most important tools in the theory of boundary value problems for holomorphic functions is the Cauchy type integral (see e.g. [57, 78, 85]).

Hence, it is not surprising that this object has also been studied in the context of Clifford analysis (see e.g. [1, 2, 5, 6, 7, 17, 18, 19, 20, 21, 22, 71, 86, 107, 108, 122, 123]).

**Definition 1.3** *If  $f$  is a continuous function defined on the surface  $\Sigma$ , then the Cauchy type integral of  $f$  is the function given by*

$$\mathbb{C}_\Sigma f(\underline{x}) = \int_\Sigma E(\underline{y} - \underline{x}) \nu(\underline{y}) f(\underline{y}) d\mathcal{H}^{m-1}(\underline{y}), \quad \underline{x} \in \mathbb{R}^m \setminus \Sigma.$$

It immediately follows that  $\mathbb{C}_\Sigma f$  is a monogenic function in  $\mathbb{R}^m \setminus \Sigma$  and vanishes at infinity.

In this section, we shall spell out some important properties of the Cauchy type integral provided in [7] that will be useful in the thesis. But first we need some definitions.

Put  $d = \text{diam}(\Sigma)$ . Let  $\theta_{\underline{z}}(\epsilon) = \mathcal{H}^{m-1}(\Sigma \cap B(\underline{z}, \epsilon))$  for  $\underline{z} \in \Sigma$  and  $\epsilon > 0$ .

**Definition 1.4** *The surface  $\Sigma$  is called Ahlfors-David-regular (AD-regular) if there exists a constant  $C > 0$  such that*

$$C^{-1} \epsilon^{m-1} \leq \theta_{\underline{z}}(\epsilon) \leq C \epsilon^{m-1}$$

for all  $\underline{z} \in \Sigma$  and  $0 < \epsilon \leq d$ .

AD-regular surfaces include smooth, Liapunov and Lipschitz surfaces but also many other arbitrary subsets of  $\mathbb{R}^m$  (see [36]).

Let us denote by  $S(\Sigma)$  the set of all continuous functions  $f$  on  $\Sigma$  such that the following integrals

$$\int_{\Sigma \cap B(\underline{z}, \epsilon)} E(\underline{y} - \underline{z}) \nu(\underline{y}) (f(\underline{y}) - f(\underline{z})) d\mathcal{H}^{m-1}(\underline{y})$$

converge uniformly to zero for  $\underline{z} \in \Sigma$  as  $\epsilon \rightarrow 0$ .

For  $f \in S(\Sigma)$ , we consider the singular version of the Cauchy type integral: the so-called singular integral operator (or Hilbert transform)  $S_\Sigma f$  defined by

$$S_\Sigma f(\underline{z}) = 2 \lim_{\epsilon \rightarrow 0} \int_{\Sigma \setminus B(\underline{z}, \epsilon)} E(\underline{y} - \underline{z}) \nu(\underline{y}) (f(\underline{y}) - f(\underline{z})) d\mathcal{H}^{m-1}(\underline{y}) + f(\underline{z}), \quad \underline{z} \in \Sigma.$$

Note that for any  $f \in S(\Sigma)$ , the singular integral operator  $S_\Sigma f$  exists for all  $\underline{z} \in \Sigma$  and it defines a continuous function on  $\Sigma$ .

The modulus of continuity of a continuous function  $f$  on  $\Sigma$  will be denoted by  $\omega_f$  and is defined by

$$\omega_f(\tau) = \tau \sup_{\delta \geq \tau} \delta^{-1} \sup_{|\underline{z}_1 - \underline{z}_2| \leq \delta} |f(\underline{z}_1) - f(\underline{z}_2)|, \quad \tau \in (0, d].$$

A function  $\varphi : (0, d] \rightarrow \mathbb{R}_+$  with  $\varphi(0+) = 0$  is said to be a majorant if  $\varphi(\tau)$  is non-decreasing and  $\varphi(\tau)/\tau$  is non-increasing for  $\tau \in (0, d]$ .

Let us denote by  $H_\varphi(\Sigma)$  the set of continuous functions  $f$  on  $\Sigma$  satisfying a generalized Hölder condition, i.e.

$$|f(\underline{z}_1) - f(\underline{z}_2)| \leq C \varphi(|\underline{z}_1 - \underline{z}_2|), \quad \underline{z}_1, \underline{z}_2 \in \Sigma,$$

or equivalently

$$\omega_f(\tau) \leq C \varphi(\tau), \quad \tau \in (0, d],$$

where  $\varphi$  is a majorant and  $C$  is a positive constant.

It is evident that for  $\varphi(\tau) = \tau^\alpha$  ( $0 < \alpha \leq 1$ ),  $H_\varphi(\Sigma)$  is nothing else but the classical set of Hölder continuous functions  $C^{0,\alpha}(\Sigma)$ . If  $\alpha = 1$ , then the function  $f$  satisfies a Lipschitz condition.

A norm  $\|f\|_{H_\varphi}$  may be defined for each  $f \in H_\varphi(\Sigma)$  by putting

$$\|f\|_{H_\varphi} = \max_{\underline{z} \in \Sigma} |f(\underline{z})| + \sup_{\underline{z}_1, \underline{z}_2 \in \Sigma} \frac{|f(\underline{z}_1) - f(\underline{z}_2)|}{\varphi(|\underline{z}_1 - \underline{z}_2|)}.$$

If moreover for a majorant  $\varphi$  there exists a constant  $C > 0$  such that

$$\int_0^\epsilon \frac{\varphi(\tau)}{\tau} d\tau + \epsilon \int_\epsilon^d \frac{\varphi(\tau)}{\tau^2} d\tau \leq C \varphi(\epsilon), \quad \epsilon \in (0, d],$$

then  $\varphi$  is said to be a regular majorant (see [66]). Note that  $\varphi(\tau) = \tau^\alpha$  ( $0 < \alpha < 1$ ) is a regular majorant.

It is worth noting that if  $\Sigma$  is an AD-regular surface and  $\varphi$  is a regular majorant, then  $H_\varphi(\Sigma) \subset S(\Sigma)$ . In fact, if  $f \in H_\varphi(\Sigma)$ , then

$$\begin{aligned} & \left| \int_{\Sigma \cap B(\underline{z}, \epsilon)} E(\underline{y} - \underline{z}) \nu(\underline{y}) (f(\underline{y}) - f(\underline{z})) d\mathcal{H}^{m-1}(\underline{y}) \right| \\ & \leq 2^{\frac{m}{2}} \int_{\Sigma \cap B(\underline{z}, \epsilon)} \frac{|f(\underline{y}) - f(\underline{z})|}{|\underline{y} - \underline{z}|^{m-1}} d\mathcal{H}^{m-1}(\underline{y}) \\ & \leq C \int_{\Sigma \cap B(\underline{z}, \epsilon)} \frac{\varphi(|\underline{y} - \underline{z}|)}{|\underline{y} - \underline{z}|^{m-1}} d\mathcal{H}^{m-1}(\underline{y}) \\ & = C \int_0^\epsilon \frac{\varphi(\tau)}{\tau^{m-1}} d\theta_{\underline{z}}(\tau) \leq C \int_0^\epsilon \frac{\varphi(\tau)}{\tau} d\tau \\ & \leq C\varphi(\epsilon) \end{aligned}$$

and from this it follows that  $f \in S(\Sigma)$ .

The following results are extensions to the case of Clifford analysis of those obtained in [103, 104] for complex-valued functions (see [7, 22]).

**Theorem 1.5 (Plemelj-Sokhotski Formulae)** *Let  $\Sigma$  be an AD-regular surface and let  $f \in S(\Sigma)$ . Then  $C_\Sigma f$  has continuous limit values on  $\Sigma$  given by*

$$C_\Sigma^\pm f(\underline{z}) = \lim_{\Omega^\pm \ni \underline{x} \rightarrow \underline{z}} C_\Sigma f(\underline{x}) = \frac{1}{2} (S_\Sigma f(\underline{z}) \pm f(\underline{z})), \quad \underline{z} \in \Sigma.$$

We must remark that if moreover  $\Sigma$  is a rectifiable surface, i.e.  $\Sigma$  is the Lipschitz image of some bounded subset of  $\mathbb{R}^{m-1}$ , then  $f \in S(\Sigma)$  is also a necessary condition for the continuity up to the boundary of the function  $C_\Sigma f$  (see [22]).

**Theorem 1.6** *Let  $\Sigma$  be an AD-regular surface. Then the singular integral operator  $S_\Sigma$  is an involution on  $S(\Sigma)$ , i.e.*

$$S_\Sigma^2 f = f$$

for all  $f \in S(\Sigma)$ .

**Theorem 1.7 (Plemelj-Privalov Theorem)** *Assume that  $\Sigma$  is an AD-regular surface and let  $\varphi$  be a regular majorant. Then  $S_\Sigma$  is a bounded operator mapping  $H_\varphi(\Sigma)$  into itself.*

It is worth remarking that if  $\Sigma$  is an AD-regular surface and if  $\varphi$  is a regular majorant, then  $C_\Sigma f$  ( $f \in H_\varphi(\Sigma)$ ) has continuous limit values on  $\Sigma$ , which by Theorems 1.5 and 1.7 belong to  $H_\varphi(\Sigma)$ .





## Chapter 2

# Special monogenic series and expressions

In this chapter some special power series expansions related to the CK-extension problem for surfaces of codimension 2 and a new class of monogenic functions are introduced (see [89, 90, 92]).

### 2.1 Steering monogenic functions

The aim of this section is to present a new collection of special monogenic functions: the so-called steering monogenic functions.

Consider the biaxial splitting  $\mathbb{R}^{m+1} = \mathbb{R}^2 \oplus \mathbb{R}^{m-1}$ . In this way, for any  $x \in \mathbb{R}^{m+1}$  we may write

$$x = z + \underline{y},$$

where

$$z = x_0 + x_1 e_1 \quad \text{and} \quad \underline{y} = \sum_{j=2}^m x_j e_j.$$

By the above, we can also split the generalized Cauchy-Riemann operator  $\partial_x$  as

$$\partial_x = \partial_z + \partial_{\underline{y}},$$

the operators  $\partial_z$  and  $\partial_{\underline{y}}$  being given by

$$\partial_z = \partial_{x_0} + e_1 \partial_{x_1}, \quad \partial_{\underline{y}} = \sum_{j=2}^m e_j \partial_{x_j}.$$

Our basic assumption is the following: let  $\Phi$  denote a family of functions  $f(z)$  with values in  $\mathbb{R}_{0,1}$  which is closed under conjugation and under the action of the operator  $\partial_z$ . That is, for any  $f \in \Phi$ ,  $\bar{f} \in \Phi$  and  $\partial_z f$  may be expressed as a linear combination of elements in  $\Phi$ .

We shall consider monogenic expressions of the form

$$\sum_j f_j(z) g_j(x), \tag{2.1}$$

where each  $f_j$  belongs to  $\Phi$  and each  $g_j$  is a  $\mathbb{R}_{0,m}$ -valued function. As the elements of  $\Phi$  “steer” the functions  $g_j$  in such a way that (2.1) is monogenic, we will call the expressions (2.1) steering monogenic functions.

In what follows, the exponential, trigonometric and power functions of  $z$  will be regarded as in the complex case by making the identification  $i \rightarrow e_1$ .

This idea leads to the following special monogenic functions.

*Exponential steering monogenic functions.*

Consider

$$\exp(z)A(x) + \exp(\bar{z})B(x), \tag{2.2}$$

with  $\Phi = \{\exp(z), \exp(\bar{z})\}$ .

An easy computation shows that

$$\partial_x(\exp(z)A + \exp(\bar{z})B) = \exp(z)(\partial_z A + \partial_{\underline{y}} B) + \exp(\bar{z})(\partial_{\underline{y}} A + \partial_z B + 2B).$$

Hence, if  $A$  and  $B$  satisfy the system

$$\begin{cases} \partial_z A + \partial_{\underline{y}} B &= 0 \\ \partial_{\underline{y}} A + \partial_z B + 2B &= 0 \end{cases}$$

then (2.2) is monogenic. In particular, if  $A$  and  $B$  only depend on the variable  $\underline{y}$ , then the above system also constitutes a necessary condition for the monogenicity of (2.2), and it takes the form

$$\begin{cases} \partial_{\underline{y}} B &= 0 \\ B &= -\frac{1}{2} \partial_{\underline{y}} A. \end{cases}$$

Substitution of the second equation of the latter system into the first one yields

$$\Delta_{\underline{y}} A = \sum_{j=2}^m \partial_{x_j}^2 A = 0,$$

i.e. the function  $A(\underline{y})$  is harmonic.

Note that we have actually proved that if  $H$  is a harmonic function of  $\underline{y}$ , then

$$\exp(z) H(\underline{y}) - \frac{1}{2} \exp(\bar{z}) (\partial_{\underline{y}} H(\underline{y}))$$

is monogenic.

For instance, if  $H(\underline{y}) = x_j$  ( $j = 2, \dots, m$ ), then we get the monogenic function

$$\exp(z) x_j - \frac{1}{2} \exp(\bar{z}) e_j.$$

Taking

$$H(\underline{y}) = \frac{1}{|\underline{y}|^{m-3}}, \quad \underline{y} \in \mathbb{R}^{m-1} \setminus \{0\},$$

we can also assert that

$$\exp(z) \frac{1}{|\underline{y}|^{m-3}} + \frac{(m-3)}{2} \exp(\bar{z}) \frac{\underline{y}}{|\underline{y}|^{m-1}}$$

is monogenic.

*Trigonometric steering monogenic functions.*

Consider

$$\cos z A_1(x) + \sin z B_1(x) + \cos \bar{z} A_2(x) + \sin \bar{z} B_2(x), \quad (2.3)$$

with  $\Phi = \{\cos z, \sin z, \cos \bar{z}, \sin \bar{z}\}$ .

A direct computation then yields

$$\begin{aligned} \partial_x(\cos z A_1 + \sin z B_1 + \cos \bar{z} A_2 + \sin \bar{z} B_2) \\ = \cos z(\partial_z A_1 + \partial_{\underline{y}} A_2) + \sin z(\partial_z B_1 + \partial_{\underline{y}} B_2) \\ + \cos \bar{z}(\partial_{\underline{y}} A_1 + \partial_z A_2 + 2B_2) + \sin \bar{z}(\partial_{\underline{y}} B_1 + \partial_z B_2 - 2A_2). \end{aligned}$$

Consequently, if

$$\begin{cases} \partial_z A_1 + \partial_{\underline{y}} A_2 & = 0 \\ \partial_z B_1 + \partial_{\underline{y}} B_2 & = 0 \\ \partial_{\underline{y}} A_1 + \partial_z A_2 + 2B_2 & = 0 \\ \partial_{\underline{y}} B_1 + \partial_z B_2 - 2A_2 & = 0 \end{cases}$$

then (2.3) is a monogenic function.

Similarly to the exponential steering monogenic functions, if  $A_k$  and  $B_k$  ( $k = 1, 2$ ) are functions of  $\underline{y}$ , then (2.3) is monogenic if and only if  $A_1$  and  $B_1$  are harmonic functions of  $\underline{y}$  and

$$A_2 = \frac{1}{2} \partial_{\underline{y}} B_1, \quad B_2 = -\frac{1}{2} \partial_{\underline{y}} A_1.$$

This gives rise to monogenic functions of the form

$$\cos z H_1(\underline{y}) + \sin z H_2(\underline{y}) + \frac{1}{2} \cos \bar{z} (\partial_{\underline{y}} H_2(\underline{y})) - \frac{1}{2} \sin \bar{z} (\partial_{\underline{y}} H_1(\underline{y})),$$

where  $H_1$  and  $H_2$  are harmonic functions of  $\underline{y}$ .

*Power steering monogenic functions.*

Consider

$$A_0(x) + \sum_{k=1}^{\infty} \left( z^k A_k(x) + \bar{z}^k B_k(x) \right), \quad (2.4)$$

with  $\Phi = \{z^k, \bar{z}^k : k \in \mathbb{N}_0\}$ .

It is easily seen that, if

$$\partial_x A_0 + 2B_1 = 0$$

and if for  $k \geq 1$

$$\begin{cases} \partial_z A_k + \partial_{\underline{y}} B_k & = 0 \\ \partial_{\underline{y}} A_k + \partial_z B_k + 2(k+1)B_{k+1} & = 0 \end{cases}$$

then (2.4) is monogenic.

In particular, if the coefficients in the series (2.4) are functions of  $\underline{y}$  only, we conclude that  $A_k$  ( $k \geq 0$ ) are harmonic functions of  $\underline{y}$  and

$$B_k = -\frac{1}{2k} \partial_{\underline{y}} A_{k-1}, \quad k \geq 1.$$

This yields monogenic functions of the form

$$H_0(\underline{y}) + \sum_{k=1}^{\infty} \left( z^k H_k(\underline{y}) - \frac{1}{2k} \bar{z}^k (\partial_{\underline{y}} H_{k-1}(\underline{y})) \right),$$

the functions  $H_k$  ( $k \geq 0$ ) being harmonic of the variable  $\underline{y}$ .

*Mixed steering monogenic functions.*

We can also consider combinations of the previous cases. For example,

$$\sum_{k=0}^{\infty} \left( z^k \exp(z) A_k(\underline{y}) + \bar{z}^k \exp(\bar{z}) B_k(\underline{y}) \right). \quad (2.5)$$

It is a simple matter to check that (2.5) is monogenic if and only if for  $k \geq 0$

$$\begin{cases} \partial_{\underline{y}} B_k & = 0 \\ \partial_{\underline{y}} A_k + 2(k+1)B_{k+1} + 2B_k & = 0. \end{cases}$$

From the above it follows that each function  $A_k$  is harmonic and the functions  $B_k$  satisfy the recurrence relation

$$B_{k+1} = -\frac{1}{2(k+1)} \left( \partial_{\underline{y}} A_k + 2B_k \right),$$

with  $B_0$  a given monogenic function of  $\underline{y}$ .

## 2.2 Monogenic power series of axial and biaxial type: toroidal expansions

We first start with a series expansion around the sphere  $S^{m-1}$  of codimension two in  $\mathbb{R}^{m+1}$  in which axial symmetry plays a central role.

In what follows, a convergent series of the form

$$S(x) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} Z^k \bar{Z}^l A_{k,l}(x), \quad Z = x_0 + (r-1)\underline{\omega}, \quad (2.6)$$

will be called a toroidal expansion of axial type.

**Theorem 2.1** *A sufficient condition for  $S(x)$  to be monogenic is given by*

$$\begin{aligned} A_{k,l+1}(x) = & -\frac{1}{2(l+1)} \left( \partial_{x_0} A_{k,l}(x) + \underline{\omega} \left( \partial_r A_{k,l}(x) + \frac{1}{r} \Gamma_{\underline{x}} A_{l,k}(x) \right) \right. \\ & \left. + \frac{(m-1)\underline{\omega}}{2r} (A_{k,l}(x) - A_{l,k}(x)) \right), \quad k, l \geq 0. \end{aligned} \quad (2.7)$$

*Proof.* Let  $z = x_0 + (r-1)i$ . Using the zero divisors  $(1+i\underline{\omega})$  and  $(1-i\underline{\omega})$  we obtain

$$\begin{aligned} Z^k \bar{Z}^l &= Z^k \bar{Z}^l \left( \frac{(1-i\underline{\omega})}{2} + \frac{(1+i\underline{\omega})}{2} \right) \\ &= z^k \bar{z}^l \frac{(1-i\underline{\omega})}{2} + \bar{z}^k z^l \frac{(1+i\underline{\omega})}{2}. \end{aligned} \quad (2.8)$$

In the same way we can see that

$$\overline{Z}^k Z^l = \bar{z}^k z^l \frac{(1 - i\underline{\omega})}{2} + z^k \bar{z}^l \frac{(1 + i\underline{\omega})}{2}. \quad (2.9)$$

Applying (1.7), (1.8) and (1.9) we get

$$\begin{aligned} \partial_{\underline{x}} \left( Z^k \overline{Z}^l A_{k,l} \right) &= k z^{k-1} \bar{z}^l i \underline{\omega} \frac{(1 - i\underline{\omega})}{2} A_{k,l} - k \bar{z}^{k-1} z^l i \underline{\omega} \frac{(1 + i\underline{\omega})}{2} A_{k,l} \\ &\quad - l z^k \bar{z}^{l-1} i \underline{\omega} \frac{(1 - i\underline{\omega})}{2} A_{k,l} + l \bar{z}^k z^{l-1} i \underline{\omega} \frac{(1 + i\underline{\omega})}{2} A_{k,l} \\ &\quad + z^k \bar{z}^l \frac{(1 - i\underline{\omega})}{2} \underline{\omega} \partial_r A_{k,l} + \bar{z}^k z^l \frac{(1 + i\underline{\omega})}{2} \underline{\omega} \partial_r A_{k,l} \\ &\quad + z^k \bar{z}^l \frac{(1 + i\underline{\omega})}{2} \frac{\underline{\omega}}{r} \Gamma_{\underline{x}} A_{k,l} + \bar{z}^k z^l \frac{(1 - i\underline{\omega})}{2} \frac{\underline{\omega}}{r} \Gamma_{\underline{x}} A_{k,l} \\ &\quad + \frac{(m-1)i}{2r} z^k \bar{z}^l A_{k,l} - \frac{(m-1)i}{2r} \bar{z}^k z^l A_{k,l}. \end{aligned}$$

This gives

$$\begin{aligned} \partial_{\underline{x}} \left( Z^k \overline{Z}^l A_{k,l} \right) &= -k Z^{k-1} \overline{Z}^l A_{k,l} + l Z^k \overline{Z}^{l-1} A_{k,l} + Z^k \overline{Z}^l \underline{\omega} \partial_r A_{k,l} \\ &\quad + \overline{Z}^k Z^l \frac{\underline{\omega}}{r} \Gamma_{\underline{x}} A_{k,l} + \frac{(m-1)i}{2r} (z^k \bar{z}^l - \bar{z}^k z^l) A_{k,l}, \end{aligned}$$

where also

$$z^k \bar{z}^l - \bar{z}^k z^l = -i \underline{\omega} \left( Z^k \overline{Z}^l - \overline{Z}^k Z^l \right),$$

the latter following from (2.8) and (2.9).

We thus get

$$\begin{aligned} \partial_{\underline{x}} \left( Z^k \overline{Z}^l A_{k,l} \right) &= -k Z^{k-1} \overline{Z}^l A_{k,l} + l Z^k \overline{Z}^{l-1} A_{k,l} \\ &\quad + Z^k \overline{Z}^l \left( \underline{\omega} \partial_r A_{k,l} + \frac{(m-1)\underline{\omega}}{2r} A_{k,l} \right) + \overline{Z}^k Z^l \left( \frac{\underline{\omega}}{r} \Gamma_{\underline{x}} A_{k,l} - \frac{(m-1)\underline{\omega}}{2r} A_{k,l} \right). \end{aligned}$$

As

$$\partial_{x_0} \left( Z^k \bar{Z}^l A_{k,l} \right) = k Z^{k-1} \bar{Z}^l A_{k,l} + l Z^k \bar{Z}^{l-1} A_{k,l} + Z^k \bar{Z}^l \partial_{x_0} A_{k,l},$$

we have

$$\begin{aligned} \partial_x \left( Z^k \bar{Z}^l A_{k,l} \right) &= 2l Z^k \bar{Z}^{l-1} A_{k,l} \\ &\quad + Z^k \bar{Z}^l \left( \partial_{x_0} A_{k,l} + \underline{\omega} \partial_r A_{k,l} + \frac{(m-1)\underline{\omega}}{2r} A_{k,l} \right) \\ &\quad + \bar{Z}^k Z^l \left( \frac{\underline{\omega}}{r} \Gamma_{\underline{x}} A_{k,l} - \frac{(m-1)\underline{\omega}}{2r} A_{k,l} \right). \end{aligned} \quad (2.10)$$

So the action of the operator  $\partial_x$  on  $S$  is given by

$$\begin{aligned} \partial_x S &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} Z^k \bar{Z}^l \left( 2(l+1) A_{k,l+1} + \partial_{x_0} A_{k,l} \right. \\ &\quad \left. + \underline{\omega} \left( \partial_r A_{k,l} + \frac{1}{r} \Gamma_{\underline{x}} A_{l,k} \right) + \frac{(m-1)\underline{\omega}}{2r} (A_{k,l} - A_{l,k}) \right). \end{aligned}$$

Hence we may conclude that the recurrence relation (2.7) is sufficient for  $S(x)$  to be monogenic.  $\square$

Although the computations are far from trivial, we note that the toroidal expansion (2.6) generates monogenic functions. All one has to do is to start from the sequence of functions  $\{A_{k,0}\}_{k \geq 0}$  (initial condition) and calculate the functions  $A_{k,l}$  via the recurrence formula (2.7).

It is of natural interest to investigate under which conditions on the initial condition  $\{A_{k,0}\}_{k \geq 0}$  the corresponding series generated by the recurrence formula (2.7) is convergent. This question, however, is still open.

An interesting particular case of the toroidal expansion is the case where the coefficients do not depend on the variable  $x_0$  and satisfy the symmetric relation  $A_{k,l}(\underline{x}) = A_{l,k}(\underline{x})$ . With this assumption we can explicitly calculate the coefficients in (2.6). Indeed, from (2.7) we see that

$$A_{k,l}(\underline{x}) = -\frac{1}{2l} \partial_{\underline{x}} A_{k,l-1}(\underline{x}).$$



It follows that

$$A_{k,l}(\underline{x}) = \frac{(-1)^{k+l}}{2^{k+l}k!l!} \partial_{\underline{x}}^{k+l} A_{0,0}(\underline{x}).$$

Substituting the above into (2.6) gives

$$\begin{aligned} S(x) &= \sum_{k=0}^{\infty} \sum_{l=0}^k Z^{k-l} \bar{Z}^l A_{k-l,l}(\underline{x}) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \sum_{l=0}^k \frac{k!}{(k-l)!l!} Z^{k-l} \bar{Z}^l \partial_{\underline{x}}^k A_{0,0}(\underline{x}) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} (Z + \bar{Z})^k \partial_{\underline{x}}^k A_{0,0}(\underline{x}). \end{aligned}$$

Clearly,

$$S(x) = \sum_{k=0}^{\infty} \frac{(-x_0)^k}{k!} \partial_{\underline{x}}^k A_{0,0}(\underline{x}),$$

which is the classical CK-extension (1.12).

We can also solve the recurrence formula (2.7) if the coefficients satisfy the relation  $A_{k,l}(x) = (-1)^{k+l} A_{l,k}(x)$ . In this case, we obtain

$$A_{k,l}(x) = \begin{cases} -\frac{1}{2l} \partial_x A_{k,l-1}(x) & \text{for } k+l \text{ odd,} \\ -\frac{1}{2l} P_x A_{k,l-1}(x) & \text{for } k+l \text{ even,} \end{cases}$$

where the differential operator  $P_x$  is defined by

$$P_x g = \partial_{x_0} g + \underline{\omega} \partial_r g + \frac{1}{r} \Gamma_{\underline{x}}(\underline{\omega} g).$$

Therefore

$$A_{k,l}(x) = \begin{cases} \frac{(-1)^l}{2^{k+l}k!l!} \partial_x (P_x \partial_x)^{\frac{k+l-1}{2}} A_{0,0}(x) & \text{for } k+l \text{ odd,} \\ \frac{(-1)^l}{2^{k+l}k!l!} (P_x \partial_x)^{\frac{k+l}{2}} A_{0,0}(x) & \text{for } k+l \text{ even.} \end{cases}$$

We thus get

$$\begin{aligned}
S(x) &= \sum_{k=0}^{\infty} \sum_{l=0}^{2k} Z^{2k-l} \bar{Z}^l A_{2k-l,l}(x) + \sum_{k=0}^{\infty} \sum_{l=0}^{2k+1} Z^{2k+1-l} \bar{Z}^l A_{2k+1-l,l}(x) \\
&= \sum_{k=0}^{\infty} \frac{1}{2^{2k}(2k)!} \sum_{l=0}^{2k} \frac{(-1)^l (2k)!}{(2k-l)! l!} Z^{2k-l} \bar{Z}^l (P_x \partial_x)^k A_{0,0}(x) \\
&\quad + \sum_{k=0}^{\infty} \frac{1}{2^{2k+1}(2k+1)!} \sum_{l=0}^{2k+1} \frac{(-1)^l (2k+1)!}{(2k+1-l)! l!} Z^{2k+1-l} \bar{Z}^l \partial_x (P_x \partial_x)^k A_{0,0}(x) \\
&= \sum_{k=0}^{\infty} \frac{1}{2^{2k}(2k)!} (Z - \bar{Z})^{2k} (P_x \partial_x)^k A_{0,0}(x) \\
&\quad + \sum_{k=0}^{\infty} \frac{1}{2^{2k+1}(2k+1)!} (Z - \bar{Z})^{2k+1} \partial_x (P_x \partial_x)^k A_{0,0}(x)
\end{aligned}$$

and, as a result:

$$\begin{aligned}
S(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{(r-1)^{2k}}{(2k)!} (P_x \partial_x)^k A_{0,0}(x) \\
&\quad + \sum_{k=0}^{\infty} (-1)^k \frac{(r-1)^{2k+1} \underline{\omega}}{(2k+1)!} \partial_x (P_x \partial_x)^k A_{0,0}(x). \quad (2.11)
\end{aligned}$$

This expansion may be considered as a kind of CK-extension for the cylinder  $r = 1$  in  $\mathbb{R}^{m+1}$ . If moreover the initial function  $A_{0,0}(x)$  does not depend on the variable  $x_0$ , then (2.11) may be regarded as the CK-extension for the sphere  $S^{m-1}$  in  $\mathbb{R}^m$ .

Let us compute this series for three simple examples.

**Example 2.1.** Let  $A_{0,0}(x) = x_0$ . It may be easily proved by induction that

$$\begin{aligned}
(P_x \partial_x)^k A_{0,0}(x) &= C_k \frac{\underline{\omega}}{r^{2k-1}}, \\
\partial_x (P_x \partial_x)^k A_{0,0}(x) &= \frac{(2k-m)C_k}{r^{2k}},
\end{aligned}$$

where the constants  $C_k$  satisfy the recurrence relation

$$\begin{aligned} C_{k+1} &= -(2k - m)(2k - m + 1)C_k, \quad k \geq 1, \\ C_1 &= (m - 1). \end{aligned}$$

Let

$$A_k^{(1)}(x) = (-1)^k \frac{(r-1)^{2k}}{(2k)!} (P_x \partial_x)^k A_{0,0}(x)$$

and

$$A_k^{(2)}(x) = (-1)^k \frac{(r-1)^{2k+1} \omega}{(2k+1)!} \partial_x (P_x \partial_x)^k A_{0,0}(x).$$

We thus get

$$\lim_{k \rightarrow \infty} \frac{|A_{k+1}^{(1)}(x)|}{|A_k^{(1)}(x)|} = \lim_{k \rightarrow \infty} \frac{|A_{k+1}^{(2)}(x)|}{|A_k^{(2)}(x)|} = \frac{(r-1)^2}{r^2}.$$

Since  $(r-1)^2/r^2 < 1$  for  $r > 1/2$ , it follows that the series (2.11) converges pointwise for  $r > 1/2$  and converges uniformly on every compact subset of  $\{x \in \mathbb{R}^{m+1} : r > 1/2\}$ .

**Example 2.2.** Let  $A_{0,0}(x) = \underline{\omega}$ . With this choice of initial function, we obtain

$$\begin{aligned} (P_x \partial_x)^k A_{0,0}(x) &= C_k \frac{\omega}{r^{2k}}, \\ \partial_x (P_x \partial_x)^k A_{0,0}(x) &= \frac{(2k - m + 1)C_k}{r^{2k+1}}, \end{aligned}$$

where the constants  $C_k$  satisfy the recurrence relation

$$\begin{aligned} C_{k+1} &= -(2k - m + 1)(2k - m + 2)C_k, \quad k \geq 1, \\ C_1 &= -(m - 1)(m - 2). \end{aligned}$$

Similar arguments to those above show that the series (2.11) converges pointwise for  $r > 1/2$  and converges uniformly on every compact subset of  $\{\underline{x} \in \mathbb{R}^m : r > 1/2\}$ .

**Example 2.3.** Let  $A_{0,0}(x) = P_l(\underline{\omega})$ , where  $P_l(\underline{\omega})$  is the restriction of a homogeneous monogenic polynomial  $P_l(\underline{x})$  of degree  $l$  in  $\mathbb{R}^m$  to  $S^{m-1}$ . It follows that

$$\begin{aligned} (P_x \partial_x)^k A_{0,0}(x) &= \frac{C_k}{r^{2k}} P_l(\underline{\omega}), \\ \partial_x (P_x \partial_x)^k A_{0,0}(x) &= -\frac{(2k+l)C_k \underline{\omega}}{r^{2k+1}} P_l(\underline{\omega}), \end{aligned}$$

where the constants  $C_k$  satisfy the recurrence relation

$$\begin{aligned} C_{k+1} &= -(2k+l)(2k+l+1)C_k, \quad k \geq 1, \\ C_1 &= -l(l+1). \end{aligned}$$

For this initial function, we also obtain that the series (2.11) converges pointwise for  $r > 1/2$  and converges uniformly on every compact subset of  $\{\underline{x} \in \mathbb{R}^m : r > 1/2\}$ .

We now investigate the generalization of the previous theorem to the biaxially symmetric case. To that end we split up  $\mathbb{R}^m$  as  $\mathbb{R}^m = \mathbb{R}^{p_1} \oplus \mathbb{R}^{p_2}$ ,  $p_1 + p_2 = m$ , yielding

$$\underline{x} = \underline{x}^{(1)} + \underline{x}^{(2)}, \quad \underline{x}^{(1)} = \sum_{j=1}^{p_1} x_j e_j, \quad \underline{x}^{(2)} = \sum_{j=1}^{p_2} x_{p_1+j} e_{p_1+j}$$

and accordingly

$$\partial_{\underline{x}} = \partial_{\underline{x}^{(1)}} + \partial_{\underline{x}^{(2)}}, \quad \partial_{\underline{x}^{(1)}} = \sum_{j=1}^{p_1} e_j \partial_{x_j}, \quad \partial_{\underline{x}^{(2)}} = \sum_{j=1}^{p_2} e_{p_1+j} \partial_{x_{p_1+j}}.$$

Introducing spherical coordinates on  $\mathbb{R}^{p_1}$  and  $\mathbb{R}^{p_2}$  respectively, i.e.

$$\underline{x}^{(k)} = r_k \underline{\omega}_k, \quad r_k = |\underline{x}^{(k)}|, \quad \underline{\omega}_k \in S^{p_k-1}, \quad k = 1, 2$$

we thus have that

$$\partial_{\underline{x}} = \underline{\omega}_1 \left( \partial_{r_1} + \frac{1}{r_1} \Gamma_{\underline{x}^{(1)}} \right) + \underline{\omega}_2 \left( \partial_{r_2} + \frac{1}{r_2} \Gamma_{\underline{x}^{(2)}} \right)$$

where

$$\Gamma_{\underline{x}^{(k)}} = -\underline{x}^{(k)} \wedge \partial_{\underline{x}^{(k)}}, \quad k = 1, 2.$$

Similar to (1.9), we have

$$\Gamma_{\underline{x}^{(k)}}(\omega_k f) = (p_k - 1)\omega_k f - \omega_k \Gamma_{\underline{x}^{(k)}} f, \quad k = 1, 2.$$

A convergent series around  $S^{p_1-1} \times S^{p_2-1}$  is called a toroidal expansion of biaxial type if it has the form

$$S(\underline{x}) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} Z^k \bar{Z}^l A_{k,l}(\underline{x}), \quad Z = (r_1 - 1) + (r_2 - 1)\omega_1\omega_2. \quad (2.12)$$

We thus obtain the following generalization of Theorem 2.1.

**Theorem 2.2** *A sufficient condition for  $S(\underline{x})$  to be monogenic is given by*

$$\begin{aligned} A_{k+1,l}(\underline{x}) &= \frac{\omega_1}{2(k+1)} \left( \omega_1 \left( \partial_{r_1} A_{k,l}(\underline{x}) + \frac{1}{r_1} \Gamma_{\underline{x}^{(1)}} A_{l,k}(\underline{x}) \right) \right. \\ &\quad \left. + \omega_2 \left( \partial_{r_2} A_{k,l}(\underline{x}) + \frac{1}{r_2} \Gamma_{\underline{x}^{(2)}} A_{l,k}(\underline{x}) \right) \right. \\ &\quad \left. + \left( \frac{(p_1 - 1)\omega_1}{2r_1} + \frac{(p_2 - 1)\omega_2}{2r_2} \right) (A_{k,l}(\underline{x}) - A_{l,k}(\underline{x})) \right), \quad k, l \geq 0. \end{aligned} \quad (2.13)$$

*Proof.* The proof is similar to the one of Theorem 2.1. In fact, we have that

$$Z^k \bar{Z}^l = z^k \bar{z}^l \frac{(1 - i\omega_1\omega_2)}{2} + \bar{z}^k z^l \frac{(1 + i\omega_1\omega_2)}{2},$$

with  $z = (r_1 - 1) + (r_2 - 1)i$ .

Therefore

$$\begin{aligned} \partial_{\underline{x}} \left( Z^k \bar{Z}^l A_{k,l} \right) &= 2k \bar{Z}^{k-1} Z^l \omega_1 A_{k,l} \\ &\quad + \bar{Z}^k Z^l \left( \omega_1 \partial_{r_1} A_{k,l} + \omega_2 \partial_{r_2} A_{k,l} + \left( \frac{(p_1 - 1)\omega_1}{2r_1} + \frac{(p_2 - 1)\omega_2}{2r_2} \right) A_{k,l} \right) \end{aligned}$$

$$\begin{aligned}
& + Z^k \bar{Z}^l \left( \frac{\omega_1}{r_1} \Gamma_{\underline{x}^{(1)}} A_{k,l} + \frac{\omega_2}{r_2} \Gamma_{\underline{x}^{(2)}} A_{k,l} \right. \\
& \left. - \left( \frac{(p_1 - 1)\omega_1}{2r_1} + \frac{(p_2 - 1)\omega_2}{2r_2} \right) A_{k,l} \right)
\end{aligned} \tag{2.14}$$

and the action of the operator  $\partial_{\underline{x}}$  on  $S$  is given by

$$\begin{aligned}
\partial_{\underline{x}} S &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \bar{Z}^k Z^l \left( 2(k+1)\omega_1 A_{k+1,l} \right. \\
& + \omega_1 \left( \partial_{r_1} A_{k,l} + \frac{1}{r_1} \Gamma_{\underline{x}^{(1)}} A_{l,k} \right) + \omega_2 \left( \partial_{r_2} A_{k,l} + \frac{1}{r_2} \Gamma_{\underline{x}^{(2)}} A_{l,k} \right) \\
& \left. + \left( \frac{(p_1 - 1)\omega_1}{2r_1} + \frac{(p_2 - 1)\omega_2}{2r_2} \right) (A_{k,l} - A_{l,k}) \right).
\end{aligned}$$

We thus have that the recurrence relation (2.13) is sufficient for the function  $S(\underline{x})$  to be monogenic.  $\square$

Note that for the toroidal expansion of biaxial type (2.12) the sequence of functions  $\{A_{0,l}(\underline{x})\}_{l \geq 0}$  is the initial condition.

Let  $P_{\underline{x}^{(1)}}$  and  $P_{\underline{x}^{(2)}}$  be the differential operators defined by

$$P_{\underline{x}^{(k)}} g = \omega_k \partial_{r_k} g + \frac{1}{r_k} \Gamma_{\underline{x}^{(k)}} (\omega_k g), \quad k = 1, 2.$$

In a completely similar way as in the axial case, using (2.13), we obtain the following Cauchy-Kowalevski like extensions around  $S^{p_1-1}$  and  $S^{p_2-1}$  respectively.

(i)  $A_{k,l}(\underline{x}) = A_{l,k}(\underline{x})$ :

$$\begin{aligned}
S(\underline{x}) &= \sum_{k=0}^{\infty} (-1)^k \frac{(r_1 - 1)^{2k}}{(2k)!} \left( \left( P_{\underline{x}^{(1)}} - \partial_{\underline{x}^{(2)}} \right) \partial_{\underline{x}} \right)^k A_{0,0}(\underline{x}) \\
&+ \sum_{k=0}^{\infty} (-1)^k \frac{(r_1 - 1)^{2k+1} \omega_1}{(2k+1)!} \partial_{\underline{x}} \left( \left( P_{\underline{x}^{(1)}} - \partial_{\underline{x}^{(2)}} \right) \partial_{\underline{x}} \right)^k A_{0,0}(\underline{x}).
\end{aligned}$$

(ii)  $A_{k,l}(\underline{x}) = (-1)^{k+l} A_{l,k}(\underline{x})$ :

$$S(\underline{x}) = \sum_{k=0}^{\infty} \frac{(r_2 - 1)^{2k}}{(2k)!} \left( \left( \partial_{\underline{x}^{(1)}} - P_{\underline{x}^{(2)}} \right) \partial_{\underline{x}} \right)^k A_{0,0}(\underline{x}) \\ + \sum_{k=0}^{\infty} \frac{(r_2 - 1)^{2k+1} \omega_2}{(2k+1)!} \partial_{\underline{x}} \left( \left( \partial_{\underline{x}^{(1)}} - P_{\underline{x}^{(2)}} \right) \partial_{\underline{x}} \right)^k A_{0,0}(\underline{x}).$$

## 2.3 Generalized CK-extensions of codimension 2

In this section we focus on the CK-extension around special surfaces of codimension 2, more specifically: around spheres and products of spheres.

**Theorem 2.3 (CK-extension theorem for  $S^{m-1}$ )** *Let  $A_{k,0}(\underline{\omega})$  ( $k \geq 0$ ) be given functions. Then there exist unique functions  $A_{k,l}(\underline{\omega})$ ,  $k \geq 0$ ,  $l > 0$ , such that the following toroidal expansion of axial type*

$$S(x) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} Z^k \bar{Z}^l A_{k,l}(\underline{\omega})$$

*is monogenic. Moreover, those functions can be calculated using the recurrence relation*

$$A_{k,l+1} = -\frac{1}{2(l+1)} \sum_{n_1=0}^k \sum_{n_2=0}^l c_{n_1+n_2,n_2} \left( \Gamma_{\underline{x}} A_{l-n_2,k-n_1}(\underline{\omega}) \right. \\ \left. + \frac{(m-1)}{2} (A_{k-n_1,l-n_2}(\underline{\omega}) - A_{l-n_2,k-n_1}(\underline{\omega})) \right), \quad k, l \geq 0,$$

with  $c_{n_1,n_2} = (-1)^{n_2} \binom{n_1}{n_2} \underline{\omega} \left( \frac{\underline{\omega}}{2} \right)^{n_1}$ .

*Proof.* Using (2.10) and the series expansion

$$\begin{aligned} \frac{1}{r} &= \sum_{n_1=0}^{\infty} (1-r)^{n_1} = \sum_{n_1=0}^{\infty} \left(\frac{\omega}{2}\right)^{n_1} (Z - \bar{Z})^{n_1} \\ &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1} (-1)^{n_2} \binom{n_1}{n_2} \left(\frac{\omega}{2}\right)^{n_1} Z^{n_1-n_2} \bar{Z}^{n_2}, \quad (0 < r < 2), \end{aligned}$$

we obtain

$$\begin{aligned} \partial_x \left( Z^k \bar{Z}^l A_{k,l} \right) &= 2l Z^k \bar{Z}^{l-1} A_{k,l} \\ &\quad + \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1} \left( Z^{k+n_1-n_2} \bar{Z}^{l+n_2} \frac{(m-1)}{2} c_{n_1,n_2} A_{k,l} \right. \\ &\quad \left. + \bar{Z}^{k+n_2} Z^{l+n_1-n_2} c_{n_1,n_2} \left( \Gamma_{\underline{x}} A_{k,l} - \frac{(m-1)}{2} A_{k,l} \right) \right). \end{aligned}$$

It follows that

$$\begin{aligned} \partial_x S &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} Z^k \bar{Z}^l \left( 2(l+1) A_{k,l+1} + \sum_{n_1=0}^k \sum_{n_2=0}^l c_{n_1+n_2,n_2} \right. \\ &\quad \left. \times \left( \Gamma_{\underline{x}} A_{l-n_2,k-n_1} + \frac{(m-1)}{2} (A_{k-n_1,l-n_2} - A_{l-n_2,k-n_1}) \right) \right), \end{aligned}$$

which proves the theorem.  $\square$

**Theorem 2.4 (CK-extension theorem for  $S^{p_1-1} \times S^{p_2-1}$ )** Consider a toroidal expansion of biaxial type of the form

$$S(\underline{x}) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} Z^k \bar{Z}^l A_{k,l}(\underline{\omega}_1, \underline{\omega}_2),$$

where  $A_{0,l}(\underline{\omega}_1, \underline{\omega}_2)$  ( $l \geq 0$ ) are given functions. Then there exist unique functions  $A_{k,l}(\underline{\omega}_1, \underline{\omega}_2)$ ,  $k > 0$ ,  $l \geq 0$ , such that the above sum  $S(\underline{x})$  is monogenic. Moreover, those functions can be calculated using the recurrence



relation

$$\begin{aligned}
A_{k+1,l}(\underline{\omega}_1, \underline{\omega}_2) &= \frac{\underline{\omega}_1}{2(k+1)} \sum_{n_1=0}^k \sum_{n_2=0}^l \left( c_{n_1+n_2, n_2}^{(1)} \Gamma_{\underline{x}^{(1)}} A_{l-n_2, k-n_1}(\underline{\omega}_1, \underline{\omega}_2) \right. \\
&\quad + c_{n_1+n_2, n_2}^{(2)} \Gamma_{\underline{x}^{(2)}} A_{l-n_2, k-n_1}(\underline{\omega}_1, \underline{\omega}_2) \\
&\quad + \left( \frac{(p_1-1)}{2} c_{n_1+n_2, n_2}^{(1)} + \frac{(p_2-1)}{2} c_{n_1+n_2, n_2}^{(2)} \right) \\
&\quad \left. \times (A_{k-n_1, l-n_2}(\underline{\omega}_1, \underline{\omega}_2) - A_{l-n_2, k-n_1}(\underline{\omega}_1, \underline{\omega}_2)) \right), \quad k, l \geq 0
\end{aligned}$$

with

$$\begin{aligned}
c_{n_1, n_2}^{(1)} &= \left( \frac{-1}{2} \right)^{n_1} \binom{n_1}{n_2} \underline{\omega}_1 \\
c_{n_1, n_2}^{(2)} &= (-1)^{n_1+n_2} \binom{n_1}{n_2} \left( \frac{\underline{\omega}_1 \underline{\omega}_2}{2} \right)^{n_1} \underline{\omega}_2.
\end{aligned}$$

*Proof.* Using (2.14) and the series expansions

$$\begin{aligned}
\frac{1}{r_1} &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1} \left( \frac{-1}{2} \right)^{n_1} \binom{n_1}{n_2} \bar{Z}^{n_1-n_2} Z^{n_2}, \quad (0 < r_1 < 2) \\
\frac{1}{r_2} &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1} (-1)^{n_1+n_2} \binom{n_1}{n_2} \left( \frac{\underline{\omega}_1 \underline{\omega}_2}{2} \right)^{n_1} \bar{Z}^{n_1-n_2} Z^{n_2}, \quad (0 < r_2 < 2).
\end{aligned}$$

we obtain

$$\begin{aligned}
\partial_{\underline{x}} \left( Z^k \bar{Z}^l A_{k,l} \right) &= 2k \bar{Z}^{k-1} Z^l \underline{\omega}_1 A_{k,l} \\
&\quad + \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1} \left( \bar{Z}^{k+n_1-n_2} Z^{l+n_2} \left( \frac{(p_1-1)}{2} c_{n_1, n_2}^{(1)} + \frac{(p_2-1)}{2} c_{n_1, n_2}^{(2)} \right) A_{k,l} \right. \\
&\quad \left. + Z^{k+n_2} \bar{Z}^{l+n_1-n_2} \left( c_{n_1, n_2}^{(1)} \Gamma_{\underline{x}^{(1)}} A_{k,l} + c_{n_1, n_2}^{(2)} \Gamma_{\underline{x}^{(2)}} A_{k,l} \right) \right)
\end{aligned}$$

$$- \left( \frac{(p_1 - 1)}{2} c_{n_1, n_2}^{(1)} + \frac{(p_2 - 1)}{2} c_{n_1, n_2}^{(2)} \right) A_{k, l} \Bigg) \Bigg) .$$

The proof now follows easily.  $\square$

Generalized CK-extension theorems may also be obtained for more general surfaces. We end this chapter with the example of a general surface of codimension 2 which intersects the coordinate planes parallel to the  $(x_0, x_1)$ -plane transversally.

Let  $p_1 = 1$  and assume that  $\alpha(\underline{x}^{(2)})$  and  $\beta(\underline{x}^{(2)})$  are given  $\mathbb{R}$ -valued functions.

**Theorem 2.5** *Consider the convergent series*

$$S(x) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} z^k \bar{z}^l A_{k, l}(\underline{x}^{(2)}), \quad z = (x_0 - \alpha(\underline{x}^{(2)})) + (x_1 - \beta(\underline{x}^{(2)}))e_1.$$

*Sufficient for  $S(x)$  to be monogenic is the recurrence relation*

$$\begin{aligned} 2(l+1)A_{k, l+1} - (l+1) \left( \partial_{\underline{x}^{(2)}} \alpha - e_1 \partial_{\underline{x}^{(2)}} \beta \right) A_{l+1, k} \\ - (k+1) \left( \partial_{\underline{x}^{(2)}} \alpha + e_1 \partial_{\underline{x}^{(2)}} \beta \right) A_{l, k+1} + \partial_{\underline{x}^{(2)}} A_{l, k} = 0, \quad k, l \geq 0. \end{aligned} \quad (2.15)$$

*Proof.* An easy computation shows that

$$\begin{aligned} \partial_x \left( z^k \bar{z}^l A_{k, l} \right) &= 2l z^k \bar{z}^{l-1} A_{k, l} - k \bar{z}^{k-1} z^l \left( \partial_{\underline{x}^{(2)}} \alpha - e_1 \partial_{\underline{x}^{(2)}} \beta \right) A_{k, l} \\ &\quad - l \bar{z}^k z^{l-1} \left( \partial_{\underline{x}^{(2)}} \alpha + e_1 \partial_{\underline{x}^{(2)}} \beta \right) A_{k, l} + \bar{z}^k z^l \partial_{\underline{x}^{(2)}} A_{k, l}, \end{aligned}$$

from which the theorem follows.  $\square$

In particular, if  $\alpha(\underline{x}^{(2)}) = \beta(\underline{x}^{(2)}) = 0$  for all  $\underline{x}^{(2)}$ , then (2.15) takes the form

$$A_{k, l+1}(\underline{x}^{(2)}) = -\frac{1}{2(l+1)} \partial_{\underline{x}^{(2)}} A_{l, k}(\underline{x}^{(2)}), \quad k, l \geq 0.$$

Solving this recurrence relation we get

$$A_{k,l}(\underline{x}^{(2)}) = \begin{cases} (-1)^l \frac{(k-l)!}{4^l k! l!} \Delta_{\underline{x}^{(2)}}^l A_{k-l,0}(\underline{x}^{(2)}) & \text{for } k \geq l, \\ \frac{(-1)^{k+1}}{2} \frac{(l-k-1)!}{4^k k! l!} \partial_{\underline{x}^{(2)}} \Delta_{\underline{x}^{(2)}}^k A_{l-k-1,0}(\underline{x}^{(2)}) & \text{for } k < l, \end{cases} \quad (2.16)$$

where  $\Delta_{\underline{x}^{(2)}} = \sum_{j=2}^m \partial_{x_j}^2$ .

We thus have obtained the following codimension 2 generalization of the CK-extension theorem.

**Corollary 2.1** *Let  $A_{k,0}(\underline{x}^{(2)})$  ( $k \geq 0$ ) be given functions, and consider the formal series*

$$f(x) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (x_0 + x_1 e_1)^k (x_0 - x_1 e_1)^l A_{k,l}(\underline{x}^{(2)}).$$

*Then there exist unique functions  $A_{k,l}(\underline{x}^{(2)})$ ,  $k \geq 0$ ,  $l > 0$ , such that the above sum  $f$  is monogenic. Moreover, those functions can be calculated using (2.16).*



## Chapter 3

# Fueter's theorems

In this chapter we present an alternative proof for and a generalization of Fueter's theorem for monogenic functions (see [87, 88, 91]).

### 3.1 An alternative proof

Fueter's theorem is named after the Swiss mathematician R. Fueter who in his 1935-paper [55] obtained a method to generate monogenic quaternionic functions starting from a holomorphic function in the upper half of the complex plane.

More precisely, if  $f(z) = u(x, y) + iv(x, y)$  ( $z = x + iy$ ) is a holomorphic function in some open subset  $\Xi \subset \mathbb{C}^+ = \{z = x + iy \in \mathbb{C} : y > 0\}$ , then in the corresponding region, the function

$$F(q_0, \underline{q}) = \Delta \left( u(q_0, |\underline{q}|) + \frac{\underline{q}}{|\underline{q}|} v(q_0, |\underline{q}|) \right)$$

is both left and right monogenic with respect to the quaternionic Cauchy-Riemann operator

$$D = \partial_{q_0} + i\partial_{q_1} + j\partial_{q_2} + k\partial_{q_3},$$

i.e.  $DF = FD = 0$ . Here  $\underline{q} = q_1i + q_2j + q_3k$  is a pure quaternion and  $\Delta = \partial_{q_0}^2 + \partial_{q_1}^2 + \partial_{q_2}^2 + \partial_{q_3}^2$  denotes the Laplace operator in four dimensional space.

In [105] Sce extended Fueter's theorem to  $\mathbb{R}_{0,m}$  for  $m$  odd, i.e. under the same assumptions on  $f$ , he showed that the function

$$\Delta_x^{\frac{m-1}{2}} (u(x_0, r) + \underline{\omega} v(x_0, r))$$

is monogenic in  $\tilde{\Omega} = \{x \in \mathbb{R}^{m+1} : (x_0, r) \in \Xi\}$ . Using Fourier transformation, Qian proved this result for  $m$  being even (see [96] and also [73]).

In [117] Sommen generalized Sce's result as follows: if  $m$  is an odd positive integer and  $P_k(\underline{x})$  is a homogeneous monogenic polynomial of degree  $k$  in  $\mathbb{R}^m$ , then

$$\Delta_x^{k+\frac{m-1}{2}} [(u(x_0, r) + \underline{\omega} v(x_0, r)) P_k(\underline{x})] \quad (3.1)$$

is also monogenic in  $\tilde{\Omega}$ .

His proof was based on the fact that

$$(u(x_0, r) + \underline{\omega} v(x_0, r)) P_k(\underline{x})$$

may be written locally as  $\bar{\partial}_x(h(x_0, r) P_k(\underline{x}))$  for some  $\mathbb{R}$ -valued harmonic function  $h$  of  $x_0$  and  $r$ . Thus (3.1) is monogenic if and only if

$$\Delta_x^{k+\frac{m+1}{2}} (h(x_0, r) P_k(\underline{x})) = 0,$$

which is true for any  $\mathbb{R}$ -valued harmonic function  $h$  in the variables  $x_0$  and  $r$ .

The aim of this section is to provide an alternative proof of Sommen's generalization. It is a constructive proof, whence it has the advantage of allowing to compute some examples.

Let us outline our proof. First, note that this version of Fueter's theorem provides us with the axial monogenic functions of degree  $k$ , i.e.

$$\Delta_x^{k+\frac{m-1}{2}} [(u(x_0, r) + \underline{\omega} v(x_0, r)) P_k(\underline{x})] = (A(x_0, r) + \underline{\omega} B(x_0, r)) P_k(\underline{x})$$

for some  $\mathbb{R}$ -valued and continuously differentiable functions  $A$  and  $B$ .

Hence the proof consists in showing that  $A$  and  $B$  satisfy the Vekua-type system (1.11). It relies on the following two lemmata.

**Lemma 3.1** *Suppose that  $f(t_1, \dots, t_d)$  and  $g(t_1, \dots, t_d)$  are  $\mathbb{R}$ -valued infinitely differentiable functions on  $\mathbb{R}^d$  and that  $D_{t_j}$  and  $D^{t_j}$  are differential operators defined by  $D_{t_j}(0)\{f\} = D^{t_j}(0)\{f\} = f$  and for  $n \geq 1$*

$$\begin{aligned} D_{t_j}(n)\{f\} &= \left(\frac{1}{t_j} \partial_{t_j}\right)^n \{f\}, \quad j = 1, \dots, d, \\ D^{t_j}(n)\{f\} &= \partial_{t_j} \left( \frac{D^{t_j}(n-1)\{f\}}{t_j} \right), \quad j = 1, \dots, d. \end{aligned}$$

Then one has

- (i)  $\partial_{t_j}^2 D_{t_j}(n)\{f\} = D_{t_j}(n)\{\partial_{t_j}^2 f\} - 2n D_{t_j}(n+1)\{f\},$
- (ii)  $\partial_{t_j} D_{t_j}(n-1)\{f/t_j\} = D^{t_j}(n)\{f\},$
- (iii)  $D^{t_j}(n)\{\partial_{t_j} f\} = \partial_{t_j} D_{t_j}(n)\{f\},$
- (iv)  $D_{t_j}(n)\{\partial_{t_j} f\} - \partial_{t_j} D^{t_j}(n)\{f\} = 2n/t_j D^{t_j}(n)\{f\},$
- (v)  $\partial_{t_j}^2 D^{t_j}(n)\{f\} = D^{t_j}(n)\{\partial_{t_j}^2 f\} - 2n D^{t_j}(n+1)\{f\},$
- (vi)  $D_{t_j}(n)\{fg\} = \sum_{s=0}^n \binom{n}{s} D_{t_j}(n-s)\{f\} D_{t_j}(s)\{g\},$
- (vii)  $D^{t_j}(n)\{fg\} = \sum_{s=0}^n \binom{n}{s} D_{t_j}(n-s)\{f\} D^{t_j}(s)\{g\}.$

*Proof.* We prove (i) by induction. When  $n = 1$ , we have

$$\begin{aligned} \partial_{t_j}^2 D_{t_j}(1)\{f\} &= \frac{\partial_{t_j}^3 f}{t_j} - 2 \frac{\partial_{t_j}^2 f}{t_j^2} + 2 \frac{\partial_{t_j} f}{t_j^3} \\ &= D_{t_j}(1)\{\partial_{t_j}^2 f\} - 2 D_{t_j}(2)\{f\} \end{aligned}$$

as desired.

Now we proceed to show that when (i) holds for a positive integer  $n$ , then it also holds for  $n + 1$ . Indeed,

$$\begin{aligned}\partial_{t_j}^2 D_{t_j}(n+1)\{f\} &= D_{t_j}(1)\{\partial_{t_j}^2 D_{t_j}(n)\{f\}\} - 2D_{t_j}(2)\{D_{t_j}(n)\{f\}\} \\ &= D_{t_j}(1)\left\{D_{t_j}(n)\{\partial_{t_j}^2 f\} - 2n D_{t_j}(n+1)\{f\}\right\} \\ &\quad - 2D_{t_j}(n+2)\{f\} \\ &= D_{t_j}(n+1)\{\partial_{t_j}^2 f\} - 2(n+1) D_{t_j}(n+2)\{f\}.\end{aligned}$$

Statement (ii) easily follows from the definition of  $D^{t_j}(n)\{f\}$ . Next, using (ii), we obtain (iii) as

$$D^{t_j}(n)\{\partial_{t_j} f\} = \partial_{t_j} D_{t_j}(n-1)\{\partial_{t_j} f/t_j\} = \partial_{t_j} D_{t_j}(n)\{f\}.$$

To obtain (iv) we use (i) and (ii):

$$\begin{aligned}D_{t_j}(n)\{\partial_{t_j} f\} - \partial_{t_j} D^{t_j}(n)\{f\} &= D_{t_j}(n)\{\partial_{t_j} f\} - \partial_{t_j}^2 D_{t_j}(n-1)\{f/t_j\} \\ &= D_{t_j}(n)\{\partial_{t_j} f\} - D_{t_j}(n-1)\{\partial_{t_j}^2 \{f/t_j\}\} + 2(n-1) D_{t_j}(n)\{f/t_j\} \\ &= D_{t_j}(n)\{\partial_{t_j} f\} - D_{t_j}(n-1)\{D_{t_j}(1)\{\partial_{t_j} f\} - 2D_{t_j}(1)\{f/t_j\}\} \\ &\quad + 2(n-1) D_{t_j}(n)\{f/t_j\} \\ &= 2n D_{t_j}(n)\{f/t_j\} = \frac{2n}{t_j} D^{t_j}(n)\{f\}.\end{aligned}$$

From (i)-(iii) it follows that

$$\begin{aligned}\partial_{t_j}^2 D^{t_j}(n)\{f\} &= \partial_{t_j}^3 D_{t_j}(n-1)\{f/t_j\} \\ &= \partial_{t_j} D_{t_j}(n-1)\{\partial_{t_j}^2 \{f/t_j\}\} - 2(n-1)\partial_{t_j} D_{t_j}(n)\{f/t_j\} \\ &= \partial_{t_j} D_{t_j}(n)\{\partial_{t_j} f\} - 2n\partial_{t_j} D_{t_j}(n)\{f/t_j\} \\ &= D^{t_j}(n)\{\partial_{t_j}^2 f\} - 2nD^{t_j}(n+1)\{f\}.\end{aligned}$$

Finally, (vi) and (vii) may be easily proved by induction. □



**Lemma 3.2** *Let  $h(x_0, r)$  be an  $\mathbb{R}$ -valued harmonic function on  $\mathbb{R}^2$ , i.e.*

$$\partial_{x_0}^2 h + \partial_r^2 h = 0.$$

*Then*

$$\begin{aligned}\Delta_x^n(h(x_0, r)P_k(\underline{x})) &= \prod_{j=1}^n (2k + m - (2j - 1)) D_r(n)\{h(x_0, r)\} P_k(\underline{x}), \\ \Delta_x^n(h(x_0, r)\underline{\omega}P_k(\underline{x})) &= \prod_{j=1}^n (2k + m - (2j - 1)) D^r(n)\{h(x_0, r)\} \underline{\omega}P_k(\underline{x}),\end{aligned}$$

*with  $n$  a positive integer.*

*Proof.* We first prove that for any twice continuously differentiable  $\mathbb{R}$ -valued function  $g(x_0, r)$  in the variables  $x_0$  and  $r$  the following equalities hold

$$\Delta_x(gP_k) = (\partial_{x_0}^2 g + \partial_r^2 g + (2k + m - 1)D_r(1)\{g\}) P_k,$$

$$\Delta_x(g\underline{\omega}P_k) = (\partial_{x_0}^2 g + \partial_r^2 g + (2k + m - 1)D^r(1)\{g\}) \underline{\omega}P_k.$$

In fact, it follows that

$$\Delta_x \underline{\omega} = -\partial_x^2 \underline{\omega} = (m - 1)\partial_x \left(\frac{1}{r}\right) = -\frac{(m - 1)}{r^2} \underline{\omega}$$

and

$$\begin{aligned}\Delta_x g &= \partial_{x_0}^2 g + \Delta_x g = \partial_{x_0}^2 g - \partial_x(\underline{\omega}\partial_r g) \\ &= \partial_{x_0}^2 g + \partial_r^2 g + \frac{m - 1}{r} \partial_r g.\end{aligned}$$

Therefore

$$\Delta_x(gP_k) = (\Delta_x g)P_k + 2 \sum_{j=1}^m (\partial_{x_j} g)(\partial_{x_j} P_k) + g(\Delta_x P_k)$$

$$\begin{aligned}
&= \left( \partial_{x_0}^2 g + \partial_r^2 g + \frac{m-1}{r} \partial_r g \right) P_k + 2 \frac{\partial_r g}{r} E_{\underline{x}} P_k \\
&= \left( \partial_{x_0}^2 g + \partial_r^2 g + \frac{2k+m-1}{r} \partial_r g \right) P_k \\
&= (\partial_{x_0}^2 g + \partial_r^2 g + (2k+m-1) D_r(1) \{g\}) P_k
\end{aligned}$$

and

$$\begin{aligned}
\Delta_x(g \underline{\omega} P_k) &= (\Delta_{\underline{x}} \underline{\omega}) g P_k + 2 \sum_{j=1}^m (\partial_{x_j} \underline{\omega}) (\partial_{x_j} (g P_k)) + \underline{\omega} \Delta_x(g P_k) \\
&= -\frac{(m-1)}{r^2} g \underline{\omega} P_k + 2 \sum_{j=1}^m \left( \frac{e_j}{r} - \frac{x_j}{r^2} \underline{\omega} \right) \left( \frac{x_j}{r} (\partial_r g) P_k + g (\partial_{x_j} P_k) \right) \\
&\quad + \left( \partial_{x_0}^2 g + \partial_r^2 g + \frac{2k+m-1}{r} \partial_r g \right) \underline{\omega} P_k \\
&= \left( \partial_{x_0}^2 g + \partial_r^2 g + (2k+m-1) \left( \frac{\partial_r g}{r} - \frac{g}{r^2} \right) \right) \underline{\omega} P_k \\
&= (\partial_{x_0}^2 g + \partial_r^2 g + (2k+m-1) D^r(1) \{g\}) \underline{\omega} P_k.
\end{aligned}$$

The proof now follows by induction using the previous equalities together with statements (i) and (v) of Lemma 3.1.

It is clear that the lemma is true in the case  $n = 1$ . Assume that the formulae hold for a positive integer  $n$ ; we will prove them for  $n + 1$ .

We thus get

$$\begin{aligned}
\Delta_x^{n+1}(h P_k) &= \prod_{j=1}^n (2k+m-(2j-1)) \Delta_x(D_r(n) \{h\} P_k) \\
&= \prod_{j=1}^n (2k+m-(2j-1)) \\
&\quad \times (\partial_{x_0}^2 D_r(n) \{h\} + \partial_r^2 D_r(n) \{h\} + (2k+m-1) D_r(n+1) \{h\}) P_k
\end{aligned}$$

$$\begin{aligned}
&= \prod_{j=1}^n (2k + m - (2j - 1)) \\
&\quad \times (D_r(n) \{ \partial_{x_0}^2 h + \partial_r^2 h \} + (2k + m - (2n + 1)) D_r(n + 1) \{ h \}) P_k \\
&= \prod_{j=1}^{n+1} (2k + m - (2j - 1)) D_r(n + 1) \{ h \} P_k,
\end{aligned}$$

which establishes the first formula. The other one may be proved similarly.

□

We are now ready to present our alternative proof of Sommen's generalization.

*Proof.* By Lemma 3.2, we get that

$$\begin{aligned}
\Delta_x^{k+\frac{m-1}{2}} [ (u(x_0, r) + \underline{\omega} v(x_0, r)) P_k(\underline{x}) ] \\
= (2k + m - 1)!! (A(x_0, r) + \underline{\omega} B(x_0, r)) P_k(\underline{x}),
\end{aligned}$$

with

$$\begin{aligned}
A &= D_r \left( k + \frac{m-1}{2} \right) \{ u \}, \\
B &= D^r \left( k + \frac{m-1}{2} \right) \{ v \}.
\end{aligned}$$

The task is now to prove that  $A$  and  $B$  satisfy the Vekua-type system (1.11). In order to do that, it will be necessary to use the assumptions on  $u$  and  $v$  and statements (iii)-(iv) of Lemma 3.1.

Indeed,

$$\begin{aligned}
\partial_{x_0} A - \partial_r B &= D_r \left( k + \frac{m-1}{2} \right) \{ \partial_{x_0} u \} - \partial_r D^r \left( k + \frac{m-1}{2} \right) \{ v \} \\
&= D_r \left( k + \frac{m-1}{2} \right) \{ \partial_r v \} - \partial_r D^r \left( k + \frac{m-1}{2} \right) \{ v \} \\
&= \frac{2k + m - 1}{r} D^r \left( k + \frac{m-1}{2} \right) \{ v \} \\
&= \frac{2k + m - 1}{r} B
\end{aligned}$$

and

$$\begin{aligned}
\partial_{x_0} B + \partial_r A &= D^r \left( k + \frac{m-1}{2} \right) \{ \partial_{x_0} v \} + \partial_r D^r \left( k + \frac{m-1}{2} \right) \{ u \} \\
&= D^r \left( k + \frac{m-1}{2} \right) \{ \partial_{x_0} v \} + D^r \left( k + \frac{m-1}{2} \right) \{ \partial_r u \} \\
&= D^r \left( k + \frac{m-1}{2} \right) \{ \partial_{x_0} v + \partial_r u \} \\
&= 0,
\end{aligned}$$

which completes the proof.  $\square$

We conclude the section with some examples.

**Example 3.1.** Let  $f(z) = iz$ . It easily follows that

$$\begin{aligned}
D_r(n) \{ r \} &= (-1)^{n+1} \frac{(2n-3)!!}{r^{2n-1}}, \\
D^r(n) \{ x_0 \} &= (-1)^n \frac{(2n-1)!!}{r^{2n}} x_0.
\end{aligned}$$

We thus get the monogenic function

$$\left( \frac{1}{r^{2k+m-2}} + \frac{(2k+m-2)x_0 \underline{x}}{r^{2k+m}} \right) P_k(\underline{x}).$$

**Example 3.2.** Consider  $f(z) = 1/z$ . It is easy to check that

$$\begin{aligned}
D_r(n) \left\{ \frac{x_0}{x_0^2 + r^2} \right\} &= (-1)^n \frac{2^n n! x_0}{(x_0^2 + r^2)^{n+1}}, \\
D^r(n) \left\{ \frac{r}{x_0^2 + r^2} \right\} &= (-1)^n \frac{2^n n! r}{(x_0^2 + r^2)^{n+1}}.
\end{aligned}$$

With this choice of initial function, we obtain the well-known monogenic function

$$\left( \frac{\bar{x}}{|x|^{2k+m+1}} \right) P_k(\underline{x}).$$

**Example 3.3** (The Gauss-distribution in Clifford analysis). Choose  $f(z) = \exp(z^2/2)$ . It may be proved that

$$D_r(n) \left\{ \exp \left( \frac{x_0^2 - r^2}{2} \right) \right\} = (-1)^n \exp \left( \frac{x_0^2 - r^2}{2} \right),$$

$$D_r(n) \{ \cos(x_0 r) \} = \sum_{s=1}^n a_s^{(n)} \frac{x_0^s}{r^{2n-s}} \cos(x_0 r + s\pi/2),$$

$$D^r(n) \{ \sin(x_0 r) \} = \sum_{s=0}^n a_{s+1}^{(n+1)} \frac{x_0^s}{r^{2n-s}} \sin(x_0 r + s\pi/2),$$

with

$$\begin{aligned} a_1^{(n)} &= (-1)^{n+1} (2n-3)!!, \\ a_s^{(n+1)} &= -(2n-s)a_s^{(n)} + a_{s-1}^{(n)}, \quad s = 2, \dots, n, \\ a_n^{(n)} &= 1. \end{aligned}$$

By statements (vi) and (vii) of Lemma 3.1, we see that

$$\begin{aligned} D_r(n) \left\{ \exp \left( \frac{x_0^2 - r^2}{2} \right) \cos(x_0 r) \right\} \\ = \exp \left( \frac{x_0^2 - r^2}{2} \right) \sum_{s=0}^n \binom{n}{s} (-1)^{n-s} D_r(s) \{ \cos(x_0 r) \}, \end{aligned}$$

$$\begin{aligned} D^r(n) \left\{ \exp \left( \frac{x_0^2 - r^2}{2} \right) \sin(x_0 r) \right\} \\ = \exp \left( \frac{x_0^2 - r^2}{2} \right) \sum_{s=0}^n \binom{n}{s} (-1)^{n-s} D^r(s) \{ \sin(x_0 r) \}. \end{aligned}$$

Hence

$$\begin{aligned} & \exp\left(\frac{x_0^2 - r^2}{2}\right) \left( \sum_{s=0}^{k+\frac{m-1}{2}} \binom{k+\frac{m-1}{2}}{s} (-1)^{k+\frac{m-1}{2}-s} D_r(s) \{\cos(x_0 r)\} \right. \\ & \quad \left. + \underline{\omega} \sum_{s=0}^{k+\frac{m-1}{2}} \binom{k+\frac{m-1}{2}}{s} (-1)^{k+\frac{m-1}{2}-s} D^r(s) \{\sin(x_0 r)\} \right) P_k(\underline{x}) \quad (3.2) \end{aligned}$$

is a monogenic function.

Note that for  $k = 0$  the restriction of the function (3.2) to  $x_0 = 0$  is

$$(-1)^{\frac{m-1}{2}} \exp(-|\underline{x}|^2/2).$$

Therefore, for  $k = 0$ , (3.2) equals, up to a multiplicative constant, the CK-extension of  $\exp(-|\underline{x}|^2/2)$ . We thus have obtained a closed formula for the CK-extension of the Gauss-distribution in  $\mathbb{R}^m$ .

For the particular case  $k = 0$  and  $m = 3$ , the function (3.2) is equal to

$$\begin{aligned} & \exp\left(\frac{x_0^2 - r^2}{2}\right) \left( \cos(x_0 r) + \frac{x_0}{r} \sin(x_0 r) \right. \\ & \quad \left. + \underline{\omega} \left( \sin(x_0 r) + \frac{\sin(x_0 r)}{r^2} - \frac{x_0}{r} \cos(x_0 r) \right) \right). \end{aligned}$$

The CK-extension of  $\exp(-|\underline{x}|^2/2)$  is also equal to (see [45])

$$\exp(-|\underline{x}|^2/2) \sum_{n=0}^{\infty} \frac{x_0^n}{n!} H_n(\underline{x}),$$

where the functions  $H_n(\underline{x})$  are polynomials in  $\underline{x}$  of degree  $n$  with real coefficients and satisfy the recurrence formula

$$H_{n+1}(\underline{x}) = \underline{x} H_n(\underline{x}) - \partial_{\underline{x}} H_n(\underline{x}).$$

The polynomials  $H_n(\underline{x})$  are called radial Hermite polynomials (see e.g. [31, 116]).

### 3.2 Generalized Fueter's theorem

Qian and Sommen proposed in [97] a new generalization of Fueter's theorem using monogenic vector-valued functions as initial functions instead of the usual holomorphic functions.

In this section our goal is to show that it is possible, in general starting from monogenic functions in a certain Clifford algebra, to generate monogenic functions in another Clifford algebra of higher dimension. In this way we present the most general form of Fueter's theorem obtained thus far.

Consider the decomposition  $\mathbb{R}^m = \bigoplus_{s=1}^d \mathbb{R}^{p_s}$ , where  $p_1, \dots, p_d$  are positive integers such that  $\sum_{s=1}^d p_s = m$ . For any  $\underline{x} \in \mathbb{R}^m$ , we may write

$$\underline{x} = \sum_{s=1}^d \underline{x}^{(s)}, \quad \underline{x}^{(s)} = \sum_{j=1}^{p_s} x_j^{(s)} e_j^{(s)}$$

and accordingly

$$\partial_{\underline{x}} = \sum_{s=1}^d \partial_{\underline{x}^{(s)}}, \quad \partial_{\underline{x}^{(s)}} = \sum_{j=1}^{p_s} e_j^{(s)} \partial_{x_j^{(s)}}$$

where the meaning of the notations  $x_j^{(s)}$  and  $e_j^{(s)}$  is obvious.

We will denote by  $\mathbb{R}_{0,d}$  the real Clifford algebra generated by the elements  $E_s$ ,  $s = 1, \dots, d$ , with the usual multiplication rules

$$\begin{aligned} E_s^2 &= -1, & s &= 1, \dots, d, \\ E_s E_{s'} + E_{s'} E_s &= 0, & 1 \leq s \neq s' \leq d. \end{aligned}$$

In what follows, we will consider an arbitrary but fixed function  $G$  on  $\mathbb{R}^{d+1}$  with values in  $\mathbb{R}_{0,d}$  in the variables  $y_0, y_1, \dots, y_d$ . Such a function can be written as

$$G(y_0, y_1, \dots, y_d) = \sum_B G_B(y_0, y_1, \dots, y_d) E_B$$

where  $E_B = E_{\beta_1} \cdots E_{\beta_l}$  and  $B = \{\beta_1, \dots, \beta_l\} \subset \{1, \dots, d\}$  is such that  $\beta_1 < \dots < \beta_l$ .

We will also assume that the function  $G$  is monogenic with respect to the generalized Cauchy-Riemann operator

$$\partial_{y_0} + \partial_{\underline{y}} = \partial_{y_0} + \sum_{s=1}^d E_s \partial_{y_s}$$

in some open subset  $\Xi \subset \{(y_0, y_1, \dots, y_d) \in \mathbb{R}^{d+1} : y_s > 0, s = 1, \dots, d\}$ , i.e.

$$(\partial_{y_0} + \partial_{\underline{y}})G = 0,$$

or equivalently, for each  $l = 0, \dots, d$

$$\begin{aligned} \sum_{|B|=l-1} \sum_{\substack{s=1 \\ s \notin B}}^d \partial_{y_s} G_B E_s E_B + \sum_{|B|=l} \partial_{y_0} G_B E_B \\ + \sum_{|B|=l+1} \sum_{s=1}^{l+1} (-1)^s \partial_{y_{\beta_s}} G_B E_{B \setminus \{\beta_s\}} = 0. \end{aligned} \quad (3.3)$$

We prove the following generalization of Fueter's theorem.

**Theorem 3.1** *Let  $G$  be as above. Assume  $p_s$  ( $s = 1, 2, \dots, d$ ) to be odd and let*

$$\mathbf{P}_k(\underline{x}) = \prod_{s=1}^d P_{k_s}(\underline{x}^{(s)})$$

with  $k = \sum_{s=1}^d k_s$  and  $P_{k_s}(\underline{x}^{(s)})$  a homogeneous monogenic polynomial of degree  $k_s$  in  $\mathbb{R}^{p_s}$  with values in the real Clifford algebra constructed over  $\mathbb{R}^{p_s}$ . Then

$$\Delta_x^{k + \frac{m-d}{2}} \left[ \left( \sum_B G_B(x_0, r_1, \dots, r_d) \varpi_B \right) \mathbf{P}_k(\underline{x}) \right]$$



is monogenic in  $\tilde{\Omega} = \{x \in \mathbb{R}^{m+1} : (x_0, r_1, \dots, r_d) \in \Xi\}$ . Here, for any  $B = \{\beta_1, \dots, \beta_l\} \subset \{1, \dots, d\}$  with  $\beta_1 < \dots < \beta_l$ , we have put  $\underline{\omega}_B = \underline{\omega}_{\beta_1} \cdots \underline{\omega}_{\beta_l}$  and  $\underline{\omega}_\emptyset = 1$ , where  $\underline{\omega}_s = \underline{x}^{(s)}/r_s$ , with  $r_s = |\underline{x}^{(s)}|$ ,  $s = 1, \dots, d$ .

We have divided the proof into a series of lemmata.

**Lemma 3.3 (Generalized Leibniz Rule)** *Let  $f$  and  $g$  be two Clifford algebra-valued continuously differentiable functions defined in some open set of  $\mathbb{R}^m$ . Then*

$$\begin{aligned} \partial_{\underline{x}}(fg) &= (\partial_{\underline{x}}f)g + \sum_{k=0}^m (-1)^k [f]_k (\partial_{\underline{x}}g) + 2 \sum_{k=1}^m \sum_{j=1}^m [e_j [f]_k]_{k-1} (\partial_{x_j}g) \\ &= (\partial_{\underline{x}}f)g + \sum_{k=0}^m (-1)^{k-1} [f]_k (\partial_{\underline{x}}g) + 2 \sum_{k=0}^{m-1} \sum_{j=1}^m [e_j [f]_k]_{k+1} (\partial_{x_j}g). \end{aligned}$$

*Proof.* Let us write  $f$  as  $f = \sum_{k=0}^m [f]_k$ . We then have that

$$\begin{aligned} \partial_{\underline{x}}(fg) &= \sum_{j=1}^m e_j ((\partial_{x_j}f)g + f(\partial_{x_j}g)) \\ &= (\partial_{\underline{x}}f)g + \sum_{k=0}^m \sum_{j=1}^m e_j [f]_k (\partial_{x_j}g), \end{aligned}$$

while also

$$\begin{aligned} e_j [f]_k &= (-1)^k [f]_k e_j + 2[e_j [f]_k]_{k-1} \\ &= (-1)^{k-1} [f]_k e_j + 2[e_j [f]_k]_{k+1}. \end{aligned}$$

From the above it follows that

$$\begin{aligned} \partial_{\underline{x}}(fg) &= (\partial_{\underline{x}}f)g + \sum_{k=0}^m (-1)^k [f]_k (\partial_{\underline{x}}g) + 2 \sum_{k=1}^m \sum_{j=1}^m [e_j [f]_k]_{k-1} (\partial_{x_j}g) \\ &= (\partial_{\underline{x}}f)g + \sum_{k=0}^m (-1)^{k-1} [f]_k (\partial_{\underline{x}}g) + 2 \sum_{k=0}^{m-1} \sum_{j=1}^m [e_j [f]_k]_{k+1} (\partial_{x_j}g), \end{aligned}$$

which establishes the formulae.  $\square$

**Lemma 3.4** *Let  $A_B(x_0, r_1, \dots, r_d)$  be  $\mathbb{R}$ -valued continuously differentiable functions in the variables  $x_0, r_1, \dots, r_d$ . Then the function*

$$\left( \sum_B A_B(x_0, r_1, \dots, r_d) \underline{\omega}_B \right) \mathbf{P}_k(\underline{x})$$

*is monogenic if for each  $l = 0, \dots, d$*

$$\left[ \sum_{|B|=l-1} \sum_{j=1, j \notin B}^d \partial_{r_j} A_B \underline{\omega}_j \underline{\omega}_B + \sum_{|B|=l} \partial_{x_0} A_B \underline{\omega}_B \right. \\ \left. + \sum_{|B|=l+1} \sum_{j=1}^{l+1} (-1)^j \left( \partial_{r_{\beta_j}} A_B + \frac{2k_{\beta_j} + p_{\beta_j} - 1}{r_{\beta_j}} A_B \right) \underline{\omega}_{B \setminus \{\beta_j\}} \right] \mathbf{P}_k(\underline{x}) = 0.$$

*Proof.* We first observe that

$$\begin{aligned} \partial_{\underline{x}(\beta_j)} \underline{\omega}_B &= (-1)^{j-1} \left( \partial_{\underline{x}(\beta_j)} \underline{\omega}_{\beta_j} \right) \underline{\omega}_{B \setminus \{\beta_j\}} \\ &= (-1)^j \frac{(p_{\beta_j} - 1)}{r_{\beta_j}} \underline{\omega}_{B \setminus \{\beta_j\}}, \end{aligned}$$

whence

$$\partial_{\underline{x}} \underline{\omega}_B = \sum_{j=1}^l (-1)^j \frac{(p_{\beta_j} - 1)}{r_{\beta_j}} \underline{\omega}_{B \setminus \{\beta_j\}}.$$

Applying the Leibniz rule of Lemma 3.3 yields

$$\begin{aligned} &\partial_{\underline{x}} \left( A_B \underline{\omega}_B \mathbf{P}_k(\underline{x}) \right) \\ &= \left[ \sum_{j=1}^d \partial_{r_j} A_B \underline{\omega}_j \underline{\omega}_B + A_B \sum_{j=1}^l (-1)^j \frac{(p_{\beta_j} - 1)}{r_{\beta_j}} \underline{\omega}_{B \setminus \{\beta_j\}} \right] \mathbf{P}_k(\underline{x}) \\ &\quad + (-1)^l A_B \underline{\omega}_B (\partial_{\underline{x}} \mathbf{P}_k(\underline{x})) + 2A_B \left[ \sum_{j=1}^l (-1)^j \frac{k_{\beta_j}}{r_{\beta_j}} \underline{\omega}_{B \setminus \{\beta_j\}} \right] \mathbf{P}_k(\underline{x}) \end{aligned}$$

$$\begin{aligned}
&= \left[ \sum_{j=1, j \notin B}^d \partial_{r_j} A_B \omega_j \omega_B \right. \\
&\quad \left. + \sum_{j=1}^l (-1)^j \left( \partial_{r_{\beta_j}} A_B + \frac{2k_{\beta_j} + p_{\beta_j} - 1}{r_{\beta_j}} A_B \right) \omega_{B \setminus \{\beta_j\}} \right] \mathbf{P}_k(\underline{x}).
\end{aligned}$$

Hence

$$\begin{aligned}
&\partial_x \left[ \left( \sum_B A_B \omega_B \right) \mathbf{P}_k(\underline{x}) \right] \\
&= \left[ \sum_B \left( \sum_{j=1, j \notin B}^d \partial_{r_j} A_B \omega_j \omega_B + \partial_{x_0} A_B \omega_B \right. \right. \\
&\quad \left. \left. + \sum_{j=1}^l (-1)^j \left( \partial_{r_{\beta_j}} A_B + \frac{2k_{\beta_j} + p_{\beta_j} - 1}{r_{\beta_j}} A_B \right) \omega_{B \setminus \{\beta_j\}} \right) \right] \mathbf{P}_k(\underline{x}) \\
&= \sum_{l=0}^d \left[ \sum_{|B|=l-1} \sum_{j=1, j \notin B}^d \partial_{r_j} A_B \omega_j \omega_B + \sum_{|B|=l} \partial_{x_0} A_B \omega_B \right. \\
&\quad \left. + \sum_{|B|=l+1} \sum_{j=1}^{l+1} (-1)^j \left( \partial_{r_{\beta_j}} A_B + \frac{2k_{\beta_j} + p_{\beta_j} - 1}{r_{\beta_j}} A_B \right) \omega_{B \setminus \{\beta_j\}} \right] \mathbf{P}_k(\underline{x}),
\end{aligned}$$

which proves the lemma.  $\square$

**Lemma 3.5** *If  $h(x_0, r_1, \dots, r_d)$  is an  $\mathbb{R}$ -valued harmonic function in the variables  $x_0, r_1, \dots, r_d$ , then*

$$\partial_{x_0}^2 \prod_{\substack{s=1 \\ s \notin B}}^d D_{r_s}(n_s) \prod_{c=1}^l D^{r_{\beta_c}}(n_{\beta_c}) \{h\} + \sum_{j=1}^d \partial_{r_j}^2 \prod_{\substack{s=1 \\ s \notin B}}^d D_{r_s}(n_s) \prod_{c=1}^l D^{r_{\beta_c}}(n_{\beta_c}) \{h\}$$

$$\begin{aligned}
&= -2 \sum_{\substack{j=1 \\ j \notin B}}^d n_j \prod_{\substack{s=1 \\ s \notin B \cup \{j\}}}^d D_{r_s}(n_s) \prod_{c=1}^l D^{r\beta_c}(n_{\beta_c}) D_{r_j}(n_j + 1) \{h\} \\
&\quad - 2 \sum_{j=1}^l n_{\beta_j} \prod_{\substack{s=1 \\ s \notin B}}^d D_{r_s}(n_s) \prod_{\substack{c=1 \\ c \neq j}}^l D^{r\beta_c}(n_{\beta_c}) D^{r\beta_j}(n_{\beta_j} + 1) \{h\}.
\end{aligned}$$

*Proof.* From the statements (i) and (v) of Lemma 3.1, it follows that

$$\begin{aligned}
&\partial_{x_0}^2 \prod_{\substack{s=1 \\ s \notin B}}^d D_{r_s}(n_s) \prod_{c=1}^l D^{r\beta_c}(n_{\beta_c}) \{h\} + \sum_{j=1}^d \partial_{r_j}^2 \prod_{\substack{s=1 \\ s \notin B}}^d D_{r_s}(n_s) \prod_{c=1}^l D^{r\beta_c}(n_{\beta_c}) \{h\} \\
&= \prod_{\substack{s=1 \\ s \notin B}}^d D_{r_s}(n_s) \prod_{c=1}^l D^{r\beta_c}(n_{\beta_c}) \{\partial_{x_0}^2 h\} \\
&\quad + \sum_{\substack{j=1 \\ j \notin B}}^d \prod_{\substack{s=1 \\ s \notin B \cup \{j\}}}^d D_{r_s}(n_s) \prod_{c=1}^l D^{r\beta_c}(n_{\beta_c}) \partial_{r_j}^2 D_{r_j}(n_j) \{h\} \\
&\quad + \sum_{j=1}^l \prod_{\substack{s=1 \\ s \notin B}}^d D_{r_s}(n_s) \prod_{\substack{c=1 \\ c \neq j}}^l D^{r\beta_c}(n_{\beta_c}) \partial_{r_{\beta_j}}^2 D^{r\beta_j}(n_{\beta_j}) \{h\} \\
&= \prod_{\substack{s=1 \\ s \notin B}}^d D_{r_s}(n_s) \prod_{c=1}^l D^{r\beta_c}(n_{\beta_c}) \{\partial_{x_0}^2 h\} \\
&\quad + \sum_{\substack{j=1 \\ j \notin B}}^d \prod_{\substack{s=1 \\ s \notin B \cup \{j\}}}^d D_{r_s}(n_s) \prod_{c=1}^l D^{r\beta_c}(n_{\beta_c}) \left\{ D_{r_j}(n_j) \{\partial_{r_j}^2 h\} - 2n_j D_{r_j}(n_j + 1) \{h\} \right\} \\
&\quad + \sum_{j=1}^l \prod_{\substack{s=1 \\ s \notin B}}^d D_{r_s}(n_s) \prod_{\substack{c=1 \\ c \neq j}}^l D^{r\beta_c}(n_{\beta_c}) \left\{ D^{r\beta_j}(n_{\beta_j}) \{\partial_{r_{\beta_j}}^2 h\} - 2n_{\beta_j} D^{r\beta_j}(n_{\beta_j} + 1) \{h\} \right\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \partial_{x_0}^2 \prod_{\substack{s=1 \\ s \notin B}}^d D_{r_s}(n_s) \prod_{c=1}^l D^{r_{\beta_c}}(n_{\beta_c}) \{h\} + \sum_{j=1}^d \partial_{r_j}^2 \prod_{\substack{s=1 \\ s \notin B}}^d D_{r_s}(n_s) \prod_{c=1}^l D^{r_{\beta_c}}(n_{\beta_c}) \{h\} \\
&= \prod_{\substack{s=1 \\ s \notin B}}^d D_{r_s}(n_s) \prod_{c=1}^l D^{r_{\beta_c}}(n_{\beta_c}) \left\{ \partial_{x_0}^2 h + \sum_{j=1}^d \partial_{r_j}^2 h \right\} \\
&\quad - 2 \sum_{\substack{j=1 \\ j \notin B}}^d n_j \prod_{\substack{s=1 \\ s \notin B \cup \{j\}}}^d D_{r_s}(n_s) \prod_{c=1}^l D^{r_{\beta_c}}(n_{\beta_c}) D_{r_j}(n_j + 1) \{h\} \\
&\quad - 2 \sum_{j=1}^l n_{\beta_j} \prod_{\substack{s=1 \\ s \notin B}}^d D_{r_s}(n_s) \prod_{\substack{c=1 \\ c \neq j}}^l D^{r_{\beta_c}}(n_{\beta_c}) D^{r_{\beta_j}}(n_{\beta_j} + 1) \{h\} \\
&= -2 \sum_{\substack{j=1 \\ j \notin B}}^d n_j \prod_{\substack{s=1 \\ s \notin B \cup \{j\}}}^d D_{r_s}(n_s) \prod_{c=1}^l D^{r_{\beta_c}}(n_{\beta_c}) D_{r_j}(n_j + 1) \{h\} \\
&\quad - 2 \sum_{j=1}^l n_{\beta_j} \prod_{\substack{s=1 \\ s \notin B}}^d D_{r_s}(n_s) \prod_{\substack{c=1 \\ c \neq j}}^l D^{r_{\beta_c}}(n_{\beta_c}) D^{r_{\beta_j}}(n_{\beta_j} + 1) \{h\},
\end{aligned}$$

which proves the statement.  $\square$

**Lemma 3.6** *Assume that  $h(x_0, r_1, \dots, r_d)$  is an  $\mathbb{R}$ -valued harmonic function in the variables  $x_0, r_1, \dots, r_d$ . Then*

$$\begin{aligned}
\Delta_x^n (h \omega_B \mathbf{P}_k(\underline{x})) &= \left( \sum \frac{n!}{n_1! \dots n_d!} \prod_{s=1}^d d_{k_s, p_s}(n_s) \prod_{\substack{s=1 \\ s \notin B}}^d D_{r_s}(n_s) \right. \\
&\quad \left. \times \prod_{c=1}^l D^{r_{\beta_c}}(n_{\beta_c}) \{h\} \right) \omega_B \mathbf{P}_k(\underline{x})
\end{aligned}$$

where the summation runs over all possible  $n_1, \dots, n_d \in \mathbb{N}_0$  such that

$$\sum_{s=1}^d n_s = n,$$

and

$$\begin{aligned} d_{k_s, p_s}(n_s) &= (2k_s + p_s - 1)(2k_s + p_s - 3) \cdots (2k_s + p_s - (2n_s - 1)) \\ d_{k_s, p_s}(0) &= 1. \end{aligned}$$

*Proof.* Suppose that  $g(x_0, r_1, \dots, r_d)$  is a twice continuously differentiable  $\mathbb{R}$ -valued function in the variables  $x_0, r_1, \dots, r_d$  and let

$$\Delta_{\underline{x}^{(s)}} = \sum_{j=1}^{p_s} \partial_{x_j^{(s)}}^2, \quad s = 1, \dots, d.$$

It follows that

$$\Delta_{\underline{x}^{(s)}}(g \underline{\omega}_B \mathbf{P}_k) = (\partial_{r_s}^2 g + (2k_s + p_s - 1)D_{r_s}(1)\{g\}) \underline{\omega}_B \mathbf{P}_k, \quad s \notin B,$$

$$\Delta_{\underline{x}^{(\beta_s)}}(g \underline{\omega}_B \mathbf{P}_k) = (\partial_{r_{\beta_s}}^2 g + (2k_{\beta_s} + p_{\beta_s} - 1)D^{r_{\beta_s}}(1)\{g\}) \underline{\omega}_B \mathbf{P}_k, \quad s = 1, \dots, l.$$

We thus get

$$\begin{aligned} \Delta_x(g \underline{\omega}_B \mathbf{P}_k) &= \left( \partial_{x_0}^2 g + \sum_{s=1}^d \partial_{r_s}^2 g + \sum_{\substack{s=1 \\ s \notin B}}^d (2k_s + p_s - 1)D_{r_s}(1)\{g\} \right. \\ &\quad \left. + \sum_{s=1}^l (2k_{\beta_s} + p_{\beta_s} - 1)D^{r_{\beta_s}}(1)\{g\} \right) \underline{\omega}_B \mathbf{P}_k. \end{aligned}$$

The proof now follows by induction using the above equality and Lemma 3.5.  $\square$

We can now prove Theorem 3.1.

*Proof.* Our proof starts with the observation that  $m - d$  is even if  $p_1, \dots, p_d$  are odd. Indeed, observe that  $m - d = \sum_{s=1}^d (p_s - 1)$ . Since each  $p_s - 1$  is even, so is  $m - d$ .

Next, by Lemma 3.6, we have

$$\begin{aligned} \Delta_x^{k + \frac{m-d}{2}} \left[ \left( \sum_B G_B(x_0, r_1, \dots, r_d) \underline{\omega}_B \right) \mathbf{P}_k(\underline{x}) \right] \\ = (2k + m - d)!! \left( \sum_B A_B(x_0, r_1, \dots, r_d) \underline{\omega}_B \right) \mathbf{P}_k(\underline{x}), \end{aligned}$$

with

$$A_B = \prod_{\substack{s=1 \\ s \notin B}}^d D_{r_s} \left( k_s + \frac{p_s - 1}{2} \right) \prod_{c=1}^l D^{r_{\beta_c}} \left( k_{\beta_c} + \frac{p_{\beta_c} - 1}{2} \right) \{G_B\}.$$

Statements (iii) and (iv) of Lemma 3.1 imply that for each  $l = 0, \dots, d$

$$\begin{aligned} \sum_{\substack{|B|=l-1 \\ j \notin B}} \sum_{\substack{j=1 \\ j \notin B}}^d \partial_{r_j} A_B \underline{\omega}_j \underline{\omega}_B &= \sum_{\substack{|B|=l-1 \\ j \notin B}} \sum_{\substack{j=1 \\ j \notin B}}^d \prod_{\substack{s=1 \\ s \notin B \cup \{j\}}}^d D_{r_s} \left( k_s + \frac{p_s - 1}{2} \right) \\ &\quad \times \prod_{c=1}^{l-1} D^{r_{\beta_c}} \left( k_{\beta_c} + \frac{p_{\beta_c} - 1}{2} \right) \partial_{r_j} D_{r_j} \left( k_j + \frac{p_j - 1}{2} \right) \{G_B\} \underline{\omega}_j \underline{\omega}_B \\ &= \sum_{\substack{|B|=l-1 \\ j \notin B}} \sum_{\substack{j=1 \\ j \notin B}}^d \prod_{\substack{s=1 \\ s \notin B \cup \{j\}}}^d D_{r_s} \left( k_s + \frac{p_s - 1}{2} \right) \\ &\quad \times \prod_{c=1}^{l-1} D^{r_{\beta_c}} \left( k_{\beta_c} + \frac{p_{\beta_c} - 1}{2} \right) D^{r_j} \left( k_j + \frac{p_j - 1}{2} \right) \{\partial_{r_j} G_B\} \underline{\omega}_j \underline{\omega}_B \end{aligned}$$

and

$$\begin{aligned}
& \sum_{|B|=l+1} \sum_{j=1}^{l+1} (-1)^j \left( \partial_{r_{\beta_j}} A_B + \frac{2k_{\beta_j} + p_{\beta_j} - 1}{r_{\beta_j}} A_B \right) \underline{\omega}_{B \setminus \{\beta_j\}} \\
&= \sum_{|B|=l+1} \sum_{j=1}^{l+1} (-1)^j \left( \prod_{s=1, s \notin B}^d D_{r_s} \left( k_s + \frac{p_s - 1}{2} \right) \right. \\
&\quad \times \prod_{\substack{c=1 \\ c \neq j}}^{l+1} D^{r_{\beta_c}} \left( k_{\beta_c} + \frac{p_{\beta_c} - 1}{2} \right) \left\{ \partial_{r_{\beta_j}} D^{r_{\beta_j}} \left( k_{\beta_j} + \frac{p_{\beta_j} - 1}{2} \right) \{G_B\} \right. \\
&\quad \left. \left. + \frac{(2k_{\beta_j} + p_{\beta_j} - 1)}{r_{\beta_j}} D^{r_{\beta_j}} \left( k_{\beta_j} + \frac{p_{\beta_j} - 1}{2} \right) \{G_B\} \right\} \right) \underline{\omega}_{B \setminus \{\beta_j\}} \\
&= \sum_{|B|=l+1} \sum_{j=1}^{l+1} (-1)^j \prod_{\substack{s=1 \\ s \notin B \setminus \{\beta_j\}}}^d D_{r_s} \left( k_s + \frac{p_s - 1}{2} \right) \\
&\quad \times \prod_{\substack{c=1 \\ c \neq j}}^{l+1} D^{r_{\beta_c}} \left( k_{\beta_c} + \frac{p_{\beta_c} - 1}{2} \right) \{ \partial_{r_{\beta_j}} G_B \} \underline{\omega}_{B \setminus \{\beta_j\}}.
\end{aligned}$$

From the above it follows that for each  $l = 0, \dots, d$

$$\begin{aligned}
& \left[ \sum_{|B|=l-1} \sum_{j=1, j \notin B}^d \partial_{r_j} A_B \underline{\omega}_j \underline{\omega}_B + \sum_{|B|=l} \partial_{x_0} A_B \underline{\omega}_B \right. \\
& \quad \left. + \sum_{|B|=l+1} \sum_{j=1}^{l+1} (-1)^j \left( \partial_{r_{\beta_j}} A_B + \frac{2k_{\beta_j} + p_{\beta_j} - 1}{r_{\beta_j}} A_B \right) \underline{\omega}_{B \setminus \{\beta_j\}} \right] \mathbf{P}_k(\underline{x}) \\
&= \frac{1}{(2k + m - d)!!} \Delta_x^{k + \frac{m-d}{2}} \left[ \left( \sum_{|B|=l-1} \sum_{\substack{j=1 \\ j \notin B}}^d \partial_{r_j} G_B \underline{\omega}_j \underline{\omega}_B + \sum_{|B|=l} \partial_{x_0} G_B \underline{\omega}_B \right) \right]
\end{aligned}$$



$$\begin{aligned}
& + \sum_{|B|=l+1} \sum_{j=1}^{l+1} (-1)^j \partial_{r_{\beta_j}} G_B \omega_{B \setminus \{\beta_j\}} \Big) \mathbf{P}_k(\underline{x}) \Big] \\
& = 0,
\end{aligned}$$

where the last equality is a consequence of (3.3). The theorem immediately follows from Lemma 3.4.  $\square$

It is worth pointing out that the conclusion of Theorem 3.1 does not hold in general if some of the integers  $p_s$  are even. For instance, for  $d = p_1 = p_2 = 2$ ,  $k = 0$  and  $G(y_0, y_1, y_2) = y_1 + y_2 - y_0 E_1 - y_0 E_2$ , we have

$$\partial_x \Delta_x (r_1 + r_2 - x_0 \omega_1 - x_0 \omega_2) = x_0 \left( \frac{1}{r_1^3} + \frac{1}{r_2^3} \right) \neq 0.$$

For this generalized version of Fueter's theorem, we also compute some examples.

**Example 3.4.** Let

$$G(y_0, y_1, \dots, y_d) = \prod_{s=1}^d y_s - y_0 \sum_{j=1}^d \prod_{\substack{s=1 \\ s \neq j}}^d y_s E_j.$$

It is easy to check that

$$\begin{aligned}
& \prod_{s=1}^d D_{r_s} \left( k_s + \frac{p_s - 1}{2} \right) \left\{ \prod_{s=1}^d r_s \right\} = (-1)^{k + \frac{m+d}{2}} \prod_{s=1}^d \frac{(2k_s + p_s - 4)!!}{r_s^{2k_s + p_s - 2}}, \\
& \prod_{\substack{s=1 \\ s \neq j}}^d D_{r_s} \left( k_s + \frac{p_s - 1}{2} \right) D^{r_j} \left( k_j + \frac{p_j - 1}{2} \right) \left\{ \prod_{\substack{s=1 \\ s \neq j}}^d r_s \right\} \\
& = (-1)^{k + \frac{m+d}{2} - 1} \frac{(2k_j + p_j - 2)!!}{r_j^{2k_j + p_j - 1}} \prod_{\substack{s=1 \\ s \neq j}}^d \frac{(2k_s + p_s - 4)!!}{r_s^{2k_s + p_s - 2}}.
\end{aligned}$$

Therefore

$$\begin{aligned} & \frac{1}{C} \Delta_x^{k+\frac{m-d}{2}} \left[ \left( \prod_{s=1}^d r_s - x_0 \sum_{j=1}^d \prod_{\substack{s=1 \\ s \neq j}}^d r_s \omega_j \right) \mathbf{P}_k(\underline{x}) \right] \\ &= \left( \prod_{s=1}^d \frac{1}{r_s^{2k_s+p_s-2}} + x_0 \sum_{j=1}^d (2k_j + p_j - 2) \prod_{\substack{s=1 \\ s \neq j}}^d \frac{1}{r_s^{2k_s+p_s-2}} \frac{\underline{x}^{(j)}}{r_j^{2k_j+p_j}} \right) \mathbf{P}_k(\underline{x}), \end{aligned}$$

where

$$C = (-1)^{k+\frac{m+d}{2}} (2k+m-d)!! \prod_{s=1}^d (2k_s + p_s - 4)!!$$

is a monogenic function.

**Example 3.5.** Consider the fundamental solution

$$G(y_0, y_1, \dots, y_d) = \frac{y_0 - \sum_{j=1}^d y_j E_j}{(y_0^2 + \sum_{j=1}^d y_j^2)^{\frac{d+1}{2}}}$$

of  $\partial_{y_0} + \partial_{\underline{y}}$ .

As

$$\begin{aligned} & \prod_{s=1}^d D_{r_s} \left( k_s + \frac{p_s-1}{2} \right) \left\{ \frac{1}{(x_0^2 + \sum_{j=1}^d r_j^2)^{\frac{d+1}{2}}} \right\} \\ &= (-1)^{k+\frac{m-d}{2}} \frac{(2k+m-1)!!}{(d-1)!!} \frac{1}{(x_0^2 + \sum_{j=1}^d r_j^2)^{k+\frac{m+1}{2}}}, \\ & \prod_{\substack{s=1 \\ s \neq j}}^d D_{r_s} \left( k_s + \frac{p_s-1}{2} \right) D^{r_j} \left( k_j + \frac{p_j-1}{2} \right) \left\{ \frac{r_j}{(x_0^2 + \sum_{j=1}^d r_j^2)^{\frac{d+1}{2}}} \right\} \\ &= (-1)^{k+\frac{m-d}{2}} \frac{(2k+m-1)!!}{(d-1)!!} \frac{r_j}{(x_0^2 + \sum_{j=1}^d r_j^2)^{k+\frac{m+1}{2}}}, \end{aligned}$$

we obtain the monogenic function

$$\frac{1}{C} \Delta_x^{k+\frac{m-d}{2}} \left[ \left( \frac{\bar{x}}{|x|^{d+1}} \right) \mathbf{P}_k(\underline{x}) \right] = \left( \frac{\bar{x}}{|x|^{2k+m+1}} \right) \mathbf{P}_k(\underline{x}),$$

where

$$C = (-1)^{k+\frac{m-d}{2}} (2k+m-d)!! \frac{(2k+m-1)!!}{(d-1)!!}.$$

**Example 3.6.** Consider the steering monogenic function

$$G(y_0, y_1, \dots, y_d) = \frac{E_1}{\left( \sum_{j=2}^d y_j^2 \right)^{\frac{d-3}{2}}} - \frac{(d-3)}{2} (y_1 + y_0 E_1) \frac{\sum_{j=2}^d y_j E_j}{\left( \sum_{j=2}^d y_j^2 \right)^{\frac{d-1}{2}}}.$$

A direct computation shows that

$$\begin{aligned} & \prod_{s=2}^d D_{r_s} \left( k_s + \frac{p_s - 1}{2} \right) D^{r_1} \left( k_1 + \frac{p_1 - 1}{2} \right) \left\{ \frac{1}{\left( \sum_{j=2}^d r_j^2 \right)^{\frac{d-3}{2}}} \right\} \\ &= (-1)^{k+\frac{m-d}{2}} \frac{(2k_1 + p_1 - 2)!!}{r_1^{2k_1+p_1-1}} \\ & \quad \times \frac{(2(k - k_1) + m - p_1 - 4)!!}{(d-5)!!} \frac{1}{\left( \sum_{j=2}^d r_j^2 \right)^{k-k_1+\frac{m-p_1-2}{2}}}, \\ & \prod_{\substack{s=1 \\ s \neq j}}^d D_{r_s} \left( k_s + \frac{p_s - 1}{2} \right) D^{r_j} \left( k_j + \frac{p_j - 1}{2} \right) \left\{ \frac{r_1 r_j}{\left( \sum_{j=2}^d r_j^2 \right)^{\frac{d-1}{2}}} \right\} \\ &= (-1)^{k+\frac{m-d}{2}+1} \frac{(2k_1 + p_1 - 4)!!}{r_1^{2k_1+p_1-2}} \\ & \quad \times \frac{(2(k - k_1) + m - p_1 - 2)!!}{(d-3)!!} \frac{r_j}{\left( \sum_{j=2}^d r_j^2 \right)^{k-k_1+\frac{m-p_1}{2}}}, \end{aligned}$$

and

$$\begin{aligned}
& \prod_{\substack{s=1 \\ s \neq 1, j}}^d D_{r_s} \left( k_s + \frac{p_s - 1}{2} \right) D^{r_1} \left( k_1 + \frac{p_1 - 1}{2} \right) \\
& \quad \times D^{r_j} \left( k_j + \frac{p_j - 1}{2} \right) \left\{ \frac{r_j}{(\sum_{j=2}^d r_j^2)^{\frac{d-1}{2}}} \right\} \\
& = (-1)^{k + \frac{m-d}{2}} \frac{(2k_1 + p_1 - 2)!!}{r_1^{2k_1 + p_1 - 1}} \\
& \quad \times \frac{(2(k - k_1) + m - p_1 - 2)!!}{(d - 3)!!} \frac{r_j}{(\sum_{j=2}^d r_j^2)^{k - k_1 + \frac{m - p_1}{2}}}.
\end{aligned}$$

We thus get the monogenic function

$$\begin{aligned}
& \frac{1}{C} \Delta_x^{k + \frac{m-d}{2}} \left[ \left( \frac{\omega_1}{(|\underline{x}|^2 - |\underline{x}^{(1)}|^2)^{\frac{d-3}{2}}} \right. \right. \\
& \quad \left. \left. - \frac{(d-3)}{2} (r_1 + x_0 \omega_1) \frac{\underline{x} - \underline{x}^{(1)}}{(|\underline{x}|^2 - |\underline{x}^{(1)}|^2)^{\frac{d-1}{2}}} \right) \mathbf{P}_k(\underline{x}) \right] \\
& = \left( \frac{(2k_1 + p_1 - 2)\underline{x}^{(1)}}{r_1^{2k_1 + p_1}} \frac{1}{(|\underline{x}|^2 - |\underline{x}^{(1)}|^2)^{k - k_1 + \frac{m - p_1 - 2}{2}}} \right. \\
& \quad \left. + \frac{(2(k - k_1) + m - p_1 - 2)}{2} \left( \frac{1}{r_1^{2k_1 + p_1 - 2}} - \frac{(2k_1 + p_1 - 2)x_0 \underline{x}^{(1)}}{r_1^{2k_1 + p_1}} \right) \right. \\
& \quad \left. \times \frac{(\underline{x} - \underline{x}^{(1)})}{(|\underline{x}|^2 - |\underline{x}^{(1)}|^2)^{k - k_1 + \frac{m - p_1}{2}}} \right) \mathbf{P}_k(\underline{x}),
\end{aligned}$$

where

$$C = (-1)^{k + \frac{m-d}{2}} (2k + m - d)!! (2k_1 + p_1 - 4)!! \frac{(2(k - k_1) + m - p_1 - 4)!!}{(d - 5)!!}.$$

## Chapter 4

# The jump problem for Hermitean monogenic functions

In this chapter we study the problem of finding a Hermitean monogenic function with a given jump on a given surface in  $\mathbb{R}^m$ ,  $m = 2n$ . Necessary and sufficient conditions for the solvability of this boundary value problem are obtained (see [10]).

### 4.1 Introduction

Hermitean Clifford analysis deals with the simultaneous null solutions of the orthogonal Dirac operator  $\partial_{\underline{x}}$  and its twisted counterpart  $\partial_{\underline{x}|}$ , introduced below. For a thorough treatment of this higher dimensional function theory we refer the reader to e.g. [24, 25, 27, 101, 102].

Let  $m = 2n$ . The Clifford vector  $\underline{x}$  in  $\mathbb{R}^m$  and the Dirac operator  $\partial_{\underline{x}}$

may thus be written respectively as

$$\underline{x} = \sum_{j=1}^n (x_j e_j + x_{n+j} e_{n+j})$$

and

$$\partial_{\underline{x}} = \sum_{j=1}^n (e_j \partial_{x_j} + e_{n+j} \partial_{x_{n+j}}).$$

We also introduce for each Clifford vector  $\underline{x}$  its twisted counterpart

$$\underline{x}| = \sum_{j=1}^n (x_{n+j} e_j - x_j e_{n+j}).$$

Note that  $\underline{x}|^2 = -|\underline{x}|^2 = -|\underline{x}|^2$ . Also observe that the Clifford vectors  $\underline{x}$  and  $\underline{x}|$  are orthogonal with respect to the standard Euclidean inner product, which implies that  $\underline{x}$  and  $\underline{x}|$  anticommute.

Consider the Fischer dual of the vector  $\underline{x}|$  given by

$$\partial_{\underline{x}|} = \sum_{j=1}^n (e_j \partial_{x_{n+j}} - e_{n+j} \partial_{x_j}).$$

We notice that this twisted Dirac operator also factorizes the Laplace operator, i.e.  $\Delta_{\underline{x}} = -\partial_{\underline{x}}^2$  and that  $\partial_{\underline{x}} \partial_{\underline{x}|} = -\partial_{\underline{x}|} \partial_{\underline{x}}$ . Its fundamental solution is the function

$$E|(\underline{x}) = -\frac{1}{\omega_{2n}} \frac{\underline{x}|}{|\underline{x}|^{2n}}, \quad \underline{x} \in \mathbb{R}^{2n} \setminus \{0\}.$$

**Definition 4.1** *A continuously differentiable function  $f$  in an open set  $\Omega$  of  $\mathbb{R}^{2n}$  with values in  $\mathbb{C}_{2n}$  is called a (left) Hermitean monogenic (or h-monogenic) function in  $\Omega$  if and only if it satisfies in  $\Omega$  the system*

$$\partial_{\underline{x}} f = \partial_{\underline{x}|} f = 0.$$

The aim of this chapter is to study the following jump problem for  $h$ -monogenic functions: under which conditions can we decompose a given  $f \in C(\Sigma)$  as

$$f = f^+ - f^-, \quad (4.1)$$

where  $f^\pm \in C(\Sigma)$  are extendable to  $h$ -monogenic functions  $F^\pm$  in  $\Omega^\pm$  with  $F^-(\infty) = 0$ ?

We recall that throughout the thesis we assume  $\Omega^+$  to be a simply connected bounded and open set in  $\mathbb{R}^{2n}$ ,  $\Omega^- = \mathbb{R}^{2n} \setminus \overline{\Omega^+}$ ,  $\Sigma$  is the boundary surface of  $\Omega^+$ , and  $\mathcal{H}^{2n-1}(\Sigma) < \infty$ .

It should be noticed that if this jump problem has a solution then it is unique. This may easily be proved using the Painlevé and Liouville theorems in the Clifford analysis setting (see [8, 26]).

This work is motivated by the results obtained in [3, 4] where a similar problem was studied for two-sided monogenic functions. For the case of harmonic vector fields we refer the reader to [14].

In order to solve the problem (4.1) we propose two different approaches. The first one uses an integral criterion for  $h$ -monogenicity; the second one is based on a new conservation law for  $h$ -monogenic functions.

## 4.2 Integral criterion for $h$ -monogenicity

In this section we require  $\Sigma$  to be an AD-regular surface. We also assume that  $f$  belongs to a generalized Hölder space  $H_\varphi(\Sigma)$ , where  $\varphi$  is a regular majorant.

Let us consider the twisted version  $C_\Sigma|f$  of the Cauchy type integral and its singular version  $S_\Sigma|f$ , defined as:

$$C_\Sigma|f(\underline{x}) = \int_\Sigma E|(\underline{y} - \underline{x})\underline{\nu}|(\underline{y})f(\underline{y}) d\mathcal{H}^{2n-1}(\underline{y}),$$

$$S_\Sigma|f(\underline{z}) = 2 \lim_{\epsilon \rightarrow 0^+} \int_{\Sigma \setminus B(\underline{z}, \epsilon)} E|(\underline{y} - \underline{z})\underline{\nu}|(\underline{y})(f(\underline{y}) - f(\underline{z})) d\mathcal{H}^{2n-1}(\underline{y}) + f(\underline{z}),$$

for  $\underline{x} \in \mathbb{R}^{2n} \setminus \Sigma$  and  $\underline{z} \in \Sigma$ .

It is easily seen that  $C_\Sigma|f$  is monogenic in  $\mathbb{R}^{2n} \setminus \Sigma$  with respect to  $\partial_{\underline{x}}$  and that  $C_\Sigma|f(\infty) = 0$ .

We now mention two important properties of these integral operators which can be derived similarly to those holding for the Cauchy type integral and its singular version given in [7]:

(a)  $S_\Sigma|f \in H_\varphi(\Sigma)$ ;

(b) for  $\underline{z} \in \Sigma$ ,

$$C_\Sigma|^\pm f(\underline{z}) = \lim_{\Omega^\pm \ni \underline{x} \rightarrow \underline{z}} C_\Sigma|f(\underline{x}) = \frac{1}{2} (S_\Sigma|f(\underline{z}) \pm f(\underline{z})).$$

**Theorem 4.1 (integral criterion)** *The function  $f$  has an  $h$ -monogenic extension  $F^\pm$  to  $\Omega^\pm$ ,  $F^-(\infty) = 0$ , if and only if  $S_\Sigma f = \pm f = S_\Sigma|f$ .*

*Proof.* Suppose that  $f$  has an  $h$ -monogenic extension  $F^+$  to  $\Omega^+$ . By Cauchy's integral formula, we have

$$C_\Sigma f(\underline{x}) = F^+(\underline{x}) = C_\Sigma|f(\underline{x}), \quad \underline{x} \in \Omega^+.$$

Theorem 1.5 and property (b) now imply that

$$S_\Sigma f = f = S_\Sigma|f. \tag{4.2}$$

Conversely, assume that  $S_\Sigma f = f = S_\Sigma|f$ . Then, from (4.2) and using once more Theorem 1.5 and property (b) we obtain

$$C_\Sigma^+ f = f = C_\Sigma|^+ f$$

Note that  $C_\Sigma f - C_\Sigma|f$  is harmonic in  $\Omega^+$  and that  $C_\Sigma^+ f - C_\Sigma|^+ f = 0$ . The maximum principle for harmonic functions now yields  $C_\Sigma f = C_\Sigma|f$  in  $\Omega^+$ , whence  $C_\Sigma f$  is  $h$ -monogenic in  $\Omega^+$ . Therefore by putting

$$F^+(\underline{x}) = \begin{cases} C_\Sigma f(\underline{x}), & \underline{x} \in \Omega^+ \\ f(\underline{x}), & \underline{x} \in \Sigma \end{cases}$$



we obtain an  $h$ -monogenic extension of  $f$  to  $\Omega^+$ . The statement for  $\Omega^-$  is proved similarly.  $\square$

We are now in the position to give a first solution to (4.1). We first claim that if  $f$  can be decomposed as in (4.1) with  $f^\pm \in H_\varphi(\Sigma)$ , then  $S_\Sigma f = S_\Sigma|f$ . Indeed, Theorem 4.1 now leads to

$$S_\Sigma f = S_\Sigma f^+ - S_\Sigma f^- = S_\Sigma|f^+ - S_\Sigma|f^- = S_\Sigma|f.$$

On the other hand, if  $S_\Sigma f = S_\Sigma|f$ , then an analysis similar to the one in the proof of Theorem 4.1 shows that  $C_\Sigma f = C_\Sigma|f$ , which implies that  $C_\Sigma f$  is  $h$ -monogenic in  $\mathbb{R}^{2n} \setminus \Sigma$ . Finally, on account of Theorems 1.5 and 1.7, and properties (a) and (b), we conclude that  $f^\pm = C_\Sigma^\pm f = C_\Sigma|^\pm f$  is a solution of the jump problem (4.1).

The above observations are summarized in the theorem below.

**Theorem 4.2** *Let  $\Sigma$  be an AD-regular surface and let  $f \in H_\varphi(\Sigma)$ , where  $\varphi$  is a regular majorant. The following statements are equivalent:*

- (i)  $f$  can be decomposed as in (4.1) with  $f^\pm \in H_\varphi(\Sigma)$ ;
- (ii)  $S_\Sigma f = S_\Sigma|f$ ;
- (iii)  $C_\Sigma f = C_\Sigma|f$ ;
- (iv)  $C_\Sigma f$  is  $h$ -monogenic in  $\mathbb{R}^{2n} \setminus \Sigma$ .

Moreover, if the jump problem (4.1) is solvable then its unique solution is given by  $f^\pm = C_\Sigma^\pm f = C_\Sigma|^\pm f$ .

### 4.3 Conservation law for $h$ -monogenic functions

In the remainder of this chapter we assume  $\Sigma$  to be a  $C^1$ -smooth surface and  $f \in C^1(\Sigma)$ . Then for  $\underline{x}$  sufficiently close to  $\Sigma$  we may assume that the

orthogonal projection of  $\underline{x}$  onto  $\Sigma$  is unique and it is denoted by  $\underline{x}_\perp$ . Let us denote by  $\underline{\nu}$  the unit normal vector on  $\Sigma$  at the point  $\underline{x}_\perp$ .

In a neighbourhood of  $\Sigma$  we have the decomposition of  $\partial_{\underline{x}}$  in its normal and its tangential parts (see [115]):

$$\partial_{\underline{x}} = -\underline{\nu}(\underline{\nu} \partial_{\underline{x}}) = \underline{\nu} \partial_{\underline{\nu}} + \partial_{\|\underline{x}\|}, \quad (4.3)$$

where

$$\partial_{\underline{\nu}} = \langle \underline{\nu}, \partial_{\underline{x}} \rangle \quad \text{and} \quad \partial_{\|\underline{x}\|} = -\underline{\nu}(\underline{\nu} \wedge \partial_{\underline{x}}).$$

Similarly,

$$\partial_{\underline{x}|} = -\underline{\nu}|(\underline{\nu}| \partial_{\underline{x}|}) = \underline{\nu}| \partial_{\underline{\nu}} + \partial_{\|\underline{x}|\|}, \quad (4.4)$$

with

$$\partial_{\|\underline{x}|\|} = -\underline{\nu}|(\underline{\nu}| \wedge \partial_{\underline{x}|}).$$

The restrictions of the operators  $\partial_{\|\underline{x}\|}$  and  $\partial_{\|\underline{x}|\|}$  to  $\Sigma$  will be denoted by  $\partial_{\underline{\omega}}$  and  $\partial_{\underline{\omega}|}$  respectively.

Let us suppose that  $F \in C^1(\overline{\Omega^+})$  is a monogenic function in  $\Omega^+$  with respect to  $\partial_{\underline{x}}$  and put  $g = F|_\Sigma$ . If moreover  $F$  is  $h$ -monogenic in  $\Omega^+$ , then from (4.3) and (4.4) we obtain that in a neighbourhood of  $\Sigma$  intersected with  $\Omega^+$

$$\begin{aligned} \partial_{\underline{\nu}} F - \underline{\nu} \partial_{\|\underline{x}\|} F &= 0, \\ \partial_{\underline{\nu}} F - \underline{\nu}| \partial_{\|\underline{x}|\|} F &= 0. \end{aligned}$$

In this way  $\underline{\nu} \partial_{\|\underline{x}\|} F = \underline{\nu}| \partial_{\|\underline{x}|\|} F$  in a neighbourhood of  $\Sigma$  intersected with  $\Omega^+$ . By continuity, we get the relation

$$\underline{\nu}| \underline{\nu} \partial_{\underline{\omega}} g + \partial_{\underline{\omega}|} g = 0 \quad (4.5)$$

on  $\Sigma$ . On the other hand, if  $g$  satisfies (4.5), then for  $G = \partial_{\underline{x}|} F$  we have

$$\begin{aligned} G &= \underline{\nu}| \partial_{\underline{\nu}} F + \partial_{\|\underline{x}|\|} F, \\ 0 &= \underline{\nu} \partial_{\underline{\nu}} F + \partial_{\|\underline{x}\|} F. \end{aligned}$$

Therefore in a neighbourhood of  $\Sigma$  intersected with  $\Omega^+$ , we obtain

$$G = \nu|_{\underline{\nu}} \partial_{\|\underline{x}\|} F + \partial_{\|\underline{x}\|} F.$$

It immediately follows that  $G|_{\Sigma} = \nu|_{\underline{\nu}} \partial_{\underline{\omega}} g + \partial_{\underline{\omega}} g = 0$ . As  $G$  is  $h$ -monogenic in  $\Omega^+$  and hence harmonic, we conclude that  $\partial_{\underline{x}} F = G = 0$  in  $\Omega^+$ .

Note that this analysis may also be applied to monogenic functions in  $\Omega^-$  with respect to  $\partial_{\underline{x}}$ , which vanish at infinity.

We thus have proved the following.

**Theorem 4.3 (conservation law)** *Let  $F^{\pm} \in C^1(\overline{\Omega^{\pm}})$  be a monogenic function in  $\Omega^{\pm}$  with respect to  $\partial_{\underline{x}}$ ,  $F^-(\infty) = 0$ . Then  $F^{\pm}$  is an  $h$ -monogenic function in  $\Omega^{\pm}$  if and only if  $g = F^{\pm}|_{\Sigma}$  satisfies (4.5).*

Let us return to the jump problem (4.1). If  $f \in C^1(\Sigma)$  can be decomposed as in (4.1) with  $f^{\pm} \in C^1(\Sigma)$ , then Theorem 4.3 now gives

$$\nu|_{\underline{\nu}} \partial_{\underline{\omega}} f + \partial_{\underline{\omega}} f = (\nu|_{\underline{\nu}} \partial_{\underline{\omega}} f^+ + \partial_{\underline{\omega}} f^+) - (\nu|_{\underline{\nu}} \partial_{\underline{\omega}} f^- + \partial_{\underline{\omega}} f^-) = 0.$$

Conversely, suppose that  $\nu|_{\underline{\nu}} \partial_{\underline{\omega}} f + \partial_{\underline{\omega}} f = 0$ . Define  $f^{\pm} = C_{\Sigma}^{\pm} f$ . We will prove that  $f^{\pm}$  is a solution of (4.1). To this end, take  $G = \partial_{\underline{x}} C_{\Sigma} f$ . It follows that

$$G = \nu|_{\underline{\nu}} \partial_{\|\underline{x}\|} C_{\Sigma} f + \partial_{\|\underline{x}\|} C_{\Sigma} f.$$

Consequently, the limit values  $G^{\pm}$  of  $G$  taken from  $\Omega^{\pm}$  are given by

$$G^{\pm} = \nu|_{\underline{\nu}} \partial_{\underline{\omega}} C_{\Sigma}^{\pm} f + \partial_{\underline{\omega}} C_{\Sigma}^{\pm} f.$$

From Theorem 1.5 we see that  $G^+ - G^- = \nu|_{\underline{\nu}} \partial_{\underline{\omega}} f + \partial_{\underline{\omega}} f = 0$ . As the function  $G$  is  $h$ -monogenic in  $\mathbb{R}^{2n} \setminus \Sigma$  and vanishes at infinity we have  $G \equiv 0$  in  $\mathbb{R}^{2n} \setminus \Sigma$ , the last equality being a consequence of the Painlevé and Liouville theorems.

We thus arrive at another characterization for the solvability of the jump problem (4.1).

**Theorem 4.4** *Let  $\Sigma$  be a  $C^1$ -smooth surface and let  $f \in C^1(\Sigma)$ . The jump problem (4.1) with  $f^{\pm} \in C^1(\Sigma)$  is solvable if and only if*

$$\nu|_{\underline{\nu}} \partial_{\underline{\omega}} f + \partial_{\underline{\omega}} f = 0.$$



## Chapter 5

# Isotonic Clifford analysis

In the first section we introduce so-called isotonic functions and we derive an integral representation formula for them (see [119]). This formula reduces on the one hand to the classical Bochner-Martinelli formula for complex-valued solutions, and on the other hand to the Bochner-Martinelli formula for biregular functions in case of real Clifford algebra-valued solutions. Section 2 is devoted to the study of the boundary values of the isotonic Cauchy type integral (see [9, 23]). In the last section we let this integral operator act on continuous  $k$ -vector valued functions which gives rise to certain Bochner-Martinelli type integrals.

### 5.1 Isotonic functions

For simplicity, but without loss of generality, we will assume that the dimension of  $\mathbb{R}^m$  is even.

Let  $m = 2n$  and denote by  $\mathbb{C}_n$  the complex Clifford algebra generated by  $(e_1, \dots, e_n)$ . Next, we introduce the primitive idempotent

$$I = \prod_{j=1}^n I_j,$$

with

$$I_j = \frac{1}{2}(1 + ie_j e_{n+j}), \quad j = 1, \dots, n.$$

The following conversion relations hold

$$e_{n+j}I = ie_jI, \quad j = 1, \dots, n, \quad (5.1)$$

and for  $a \in \mathbb{C}_n$  we also have that

$$aI = 0 \Leftrightarrow a = 0. \quad (5.2)$$

Below, we will need the following Clifford vectors and their corresponding Dirac operators:

$$\begin{aligned} \underline{x}_1 &= \sum_{j=1}^n x_j e_j, & \partial_{\underline{x}_1} &= \sum_{j=1}^n e_j \partial_{x_j} \\ \underline{x}_2 &= \sum_{j=1}^n x_{n+j} e_j, & \partial_{\underline{x}_2} &= \sum_{j=1}^n e_j \partial_{x_{n+j}}. \end{aligned}$$

Now consider two Clifford vectors  $\underline{x}, \underline{y} \in \mathbb{R}_{0,2n}$ , which may be written as

$$\underline{x} = \sum_{j=1}^n (x_j e_j + x_{n+j} e_{n+j}), \quad \underline{y} = \sum_{j=1}^n (y_j e_j + y_{n+j} e_{n+j}).$$

For  $a \in \mathbb{C}_n$  it follows that

$$\underline{x}aI = \underline{x}_1aI + \sum_{j=1}^n x_{n+j}e_{n+j}aI = \underline{x}_1aI + \tilde{a} \sum_{j=1}^n x_{n+j}e_{n+j}I,$$

whence application of (5.1) yields

$$\underline{x}aI = (\underline{x}_1a + i\tilde{a}\underline{x}_2)I. \quad (5.3)$$

From the above equality, we deduce that

$$\underline{x}\underline{y}aI = (\underline{x}_1(\underline{y}_1a + i\tilde{a}\underline{y}_2) + (a\underline{y}_2 - i\underline{y}_1\tilde{a})\underline{x}_2)I. \quad (5.4)$$

If we now take a continuously differentiable function  $f : \Omega \subset \mathbb{R}^{2n} \rightarrow \mathbb{C}_n$ , then we learn from (5.3) that

$$\partial_{\underline{x}}(fI) = \left( \partial_{\underline{x}_1} f + i\tilde{f}\partial_{\underline{x}_2} \right) I,$$

whence it follows from (5.2) that the spinor-valued function  $fI$  is monogenic if and only if (see [27, 101, 119])

$$\partial_{\underline{x}_1} f + i\tilde{f}\partial_{\underline{x}_2} = 0.$$

**Definition 5.1** *A function  $f$  defined and continuously differentiable in an open set  $\Omega$  of  $\mathbb{R}^{2n}$  with values in  $\mathbb{C}_n$ , which satisfies in  $\Omega$  the above equation, is said to be isotonic in  $\Omega$ .*

Note that an isotonic function is also harmonic. This may be easily proved using (5.2) as well as the equalities

$$0 = \Delta_{\underline{x}}(fI) = (\Delta_{\underline{x}}f)I.$$

The isotonic functions are closely related with the  $h$ -monogenic functions. Indeed,  $fI$  is  $h$ -monogenic if and only if  $[f]_k$  is isotonic for  $k = 0, \dots, n$  (see [27]).

It is worth noting that if in particular  $f$  takes values in the space of scalars  $\mathbb{C}$ , then  $f$  is isotonic if and only if

$$(\partial_{x_j} + i\partial_{x_{n+j}})f = 0, \quad j = 1, \dots, n,$$

which means that  $f$  is a holomorphic function in the complex variables  $z_j = x_j + ix_{n+j}$ ,  $j = 1, \dots, n$ . On the other hand, if  $f$  takes values in the real Clifford algebra  $\mathbb{R}_{0,n}$ , then  $f$  is isotonic if and only if

$$\partial_{\underline{x}_1} f = \tilde{f}\partial_{\underline{x}_2} = 0$$

or, equivalently, by the action of the main involution on the last equality:

$$\partial_{\underline{x}_1} f = f\partial_{\underline{x}_2} = 0.$$

**Definition 5.2** A continuously differentiable function  $f$  in an open set  $\Omega$  of  $\mathbb{R}^{2n}$  with values in  $\mathbb{R}_{0,n}$  is called biregular in  $\Omega$  if and only if it satisfies in  $\Omega$  the system

$$\partial_{\underline{x}_1} f = f \partial_{\underline{x}_2} = 0.$$

The biregular functions were introduced by Brackx and Pincket as an extension to two Clifford variables of the monogenic functions in one Clifford variable. For a detailed study we refer the reader to [28, 29, 30, 95, 113, 114].

We will now derive the basic integral formulae for the isotonic functions. For that purpose, we put

$$\nu_1(\underline{y}) = \sum_{j=1}^n \nu_j(\underline{y}) e_j \quad \text{and} \quad \nu_2(\underline{y}) = \sum_{j=1}^n \nu_{n+j}(\underline{y}) e_j, \quad \underline{y} \in \Sigma.$$

Assume that  $f$  is a  $\mathbb{C}_n$ -valued continuously differentiable function in  $\overline{\Omega^+}$ . By Borel-Pompeiu's formula, we see that

$$\begin{aligned} \int_{\Sigma} E(\underline{y} - \underline{x}) \nu(\underline{y}) f(\underline{y}) Id\mathcal{H}^{2n-1}(\underline{y}) - \int_{\Omega^+} E(\underline{y} - \underline{x}) \partial_{\underline{y}}(f(\underline{y}) I) d\mathcal{L}^{2n}(\underline{y}) \\ = \begin{cases} f(\underline{x}) I & \text{for } \underline{x} \in \Omega^+, \\ 0 & \text{for } \underline{x} \in \Omega^-. \end{cases} \end{aligned}$$

The equality (5.4) now implies

$$\begin{aligned} \int_{\Sigma} E(\underline{y} - \underline{x}) \nu(\underline{y}) f(\underline{y}) Id\mathcal{H}^{2n-1}(\underline{y}) &= \int_{\Sigma} \left( E_1(\underline{y} - \underline{x}) (\nu_1(\underline{y}) f(\underline{y}) + i \tilde{f}(\underline{y}) \nu_2(\underline{y})) \right. \\ &\quad \left. + (f(\underline{y}) \nu_2(\underline{y}) - i \nu_1(\underline{y}) \tilde{f}(\underline{y})) E_2(\underline{y} - \underline{x}) \right) Id\mathcal{H}^{2n-1}(\underline{y}), \\ \int_{\Omega^+} E(\underline{y} - \underline{x}) \partial_{\underline{y}}(f(\underline{y}) I) d\mathcal{L}^{2n}(\underline{y}) &= \int_{\Omega^+} \left( E_1(\underline{y} - \underline{x}) (\partial_{\underline{y}_1} f(\underline{y}) + i \tilde{f}(\underline{y}) \partial_{\underline{y}_2}) \right. \\ &\quad \left. + (f(\underline{y}) \partial_{\underline{y}_2} - i \partial_{\underline{y}_1} \tilde{f}(\underline{y})) E_2(\underline{y} - \underline{x}) \right) Id\mathcal{L}^{2n}(\underline{y}), \end{aligned}$$



where

$$E_1(\underline{x}) = -\frac{1}{\omega_{2n}} \frac{\underline{x}_1}{|\underline{x}|^{2n}} \quad \text{and} \quad E_2(\underline{x}) = -\frac{1}{\omega_{2n}} \frac{\underline{x}_2}{|\underline{x}|^{2n}}, \quad \underline{x} \in \mathbb{R}^{2n} \setminus \{0\}.$$

Now applying (5.2), we get the following results.

**Theorem 5.1** *If  $f$  is a  $\mathbb{C}_n$ -valued continuously differentiable function in  $\overline{\Omega^+}$ , then*

$$\begin{aligned} & \int_{\Sigma} \left( E_1(\underline{y} - \underline{x})(\nu_1(\underline{y})f(\underline{y}) + i\tilde{f}(\underline{y})\nu_2(\underline{y})) \right. \\ & \quad \left. + (f(\underline{y})\nu_2(\underline{y}) - i\nu_1(\underline{y})\tilde{f}(\underline{y}))E_2(\underline{y} - \underline{x}) \right) d\mathcal{H}^{2n-1}(\underline{y}) \\ & - \int_{\Omega^+} \left( E_1(\underline{y} - \underline{x})(\partial_{\underline{y}_1}f(\underline{y}) + i\tilde{f}(\underline{y})\partial_{\underline{y}_2}) \right. \\ & \quad \left. + (f(\underline{y})\partial_{\underline{y}_2} - i\partial_{\underline{y}_1}\tilde{f}(\underline{y}))E_2(\underline{y} - \underline{x}) \right) d\mathcal{L}^{2n}(\underline{y}) = \begin{cases} f(\underline{x}) & \text{for } \underline{x} \in \Omega^+, \\ 0 & \text{for } \underline{x} \in \Omega^-. \end{cases} \end{aligned}$$

**Theorem 5.2** *Let  $f$  be a  $\mathbb{C}_n$ -valued continuous function on  $\overline{\Omega^+}$  which moreover is isotonic in  $\Omega^+$ . Then*

$$\begin{aligned} f(\underline{x}) &= \int_{\Sigma} \left( E_1(\underline{y} - \underline{x})(\nu_1(\underline{y})f(\underline{y}) + i\tilde{f}(\underline{y})\nu_2(\underline{y})) \right. \\ & \quad \left. + (f(\underline{y})\nu_2(\underline{y}) - i\nu_1(\underline{y})\tilde{f}(\underline{y}))E_2(\underline{y} - \underline{x}) \right) d\mathcal{H}^{2n-1}(\underline{y}), \quad \underline{x} \in \Omega^+. \end{aligned}$$

Let us now mention two important consequences of the previous theorem.

**Corollary 5.1** *Suppose that  $f$  is a continuous function on  $\overline{\Omega^+}$  which moreover is holomorphic in  $\Omega^+$ . Then for  $\underline{x} \in \Omega^+$  we have that*

$$\begin{aligned} \text{(i)} \quad & f(\underline{x}) = - \int_{\Sigma} \langle E_1(\underline{y} - \underline{x}) - iE_2(\underline{y} - \underline{x}), \nu_1(\underline{y}) + i\nu_2(\underline{y}) \rangle f(\underline{y}) d\mathcal{H}^{2n-1}(\underline{y}), \\ \text{(ii)} \quad & \int_{\Sigma} (E_1(\underline{y} - \underline{x}) + iE_2(\underline{y} - \underline{x})) \wedge (\nu_1(\underline{y}) + i\nu_2(\underline{y})) f(\underline{y}) d\mathcal{H}^{2n-1}(\underline{y}) = 0. \end{aligned}$$

*Proof.* As we may assume  $f$  to be  $\mathbb{C}$ -valued, we obtain (i) and (ii) respectively as the scalar and the bivector part of the formula in Theorem 5.2.  $\square$

**Corollary 5.2** *If  $f$  is an  $\mathbb{R}_{0,n}$ -valued continuous function on  $\overline{\Omega^+}$  which moreover is biregular in  $\Omega^+$ , then for  $\underline{x} \in \Omega^+$  we have that*

$$(i) \quad f(\underline{x}) = \int_{\Sigma} E_1(\underline{y} - \underline{x}) \nu_1(\underline{y}) f(\underline{y}) + f(\underline{y}) \nu_2(\underline{y}) E_2(\underline{y} - \underline{x}) d\mathcal{H}^{2n-1}(\underline{y}),$$

$$(ii) \quad \int_{\Sigma} E_1(\underline{y} - \underline{x}) f(\underline{y}) \nu_2(\underline{y}) - \nu_1(\underline{y}) f(\underline{y}) E_2(\underline{y} - \underline{x}) d\mathcal{H}^{2n-1}(\underline{y}) = 0.$$

*Proof.* The proof easily follows by taking the real and the imaginary part of the formula in Theorem 5.2.  $\square$

Note that the first statement of Corollary 5.1 corresponds to the classical Bochner-Martinelli formula (see [80]) while the first statement of Corollary 5.2 is the Bochner-Martinelli formula for biregular functions (see [28]).

## 5.2 The isotonic Cauchy type integral

We begin by introducing the main objects of the section.

**Definition 5.3** *Let  $f$  be a  $\mathbb{C}_n$ -valued continuous function on  $\Sigma$ . The isotonic Cauchy type integral of  $f$  will be denoted by  $\mathbb{C}_{\Sigma}^{isot} f$  and defined by*

$$\begin{aligned} \mathbb{C}_{\Sigma}^{isot} f(\underline{x}) = & \int_{\Sigma} \left( E_1(\underline{y} - \underline{x}) (\nu_1(\underline{y}) f(\underline{y}) + i \tilde{f}(\underline{y}) \nu_2(\underline{y})) \right. \\ & \left. + (f(\underline{y}) \nu_2(\underline{y}) - i \nu_1(\underline{y}) \tilde{f}(\underline{y})) E_2(\underline{y} - \underline{x}) \right) d\mathcal{H}^{2n-1}(\underline{y}), \quad \underline{x} \in \mathbb{R}^{2n} \setminus \Sigma. \end{aligned}$$

Since

$$\mathbb{C}_{\Sigma}(fI) = (\mathbb{C}_{\Sigma}^{isot} f)I, \tag{5.5}$$

it follows that  $\mathbb{C}_\Sigma^{isot} f$  is isotonic in  $\mathbb{R}^{2n} \setminus \Sigma$  and vanishes at infinity.

Let us introduce the space  $S^{isot}(\Sigma)$  consisting of all  $\mathbb{C}_n$ -valued continuous functions  $f$  on  $\Sigma$  for which the integrals

$$\begin{aligned} & \int_{\Sigma \cap B(\underline{z}, \epsilon)} \left( E_1(\underline{y} - \underline{z})(\underline{\nu}_1(\underline{y})(f(\underline{y}) - f(\underline{z})) + i(\tilde{f}(\underline{y}) - \tilde{f}(\underline{z}))\underline{\nu}_2(\underline{y})) \right. \\ & \quad \left. + ((f(\underline{y}) - f(\underline{z}))\underline{\nu}_2(\underline{y}) - i\underline{\nu}_1(\underline{y})(\tilde{f}(\underline{y}) - \tilde{f}(\underline{z})))E_2(\underline{y} - \underline{z}) \right) d\mathcal{H}^{2n-1}(\underline{y}) \end{aligned}$$

converge uniformly to zero for  $\underline{z} \in \Sigma$  as  $\epsilon \rightarrow 0$ . At this point it is important to notice that  $f \in S^{isot}(\Sigma)$  if and only if  $fI \in S(\Sigma)$ .

**Definition 5.4** For  $f \in S^{isot}(\Sigma)$  and  $\underline{z} \in \Sigma$ , we define the isotonic singular integral operator of  $f$  as

$$\mathbb{S}_\Sigma^{isot} f(\underline{z}) = 2 \lim_{\epsilon \rightarrow 0^+} \mathbb{S}_{\Sigma, \epsilon}^{isot} f(\underline{z}) + f(\underline{z}),$$

where  $\mathbb{S}_{\Sigma, \epsilon}^{isot} f$  denotes the truncated integral defined by

$$\begin{aligned} \mathbb{S}_{\Sigma, \epsilon}^{isot} f(\underline{z}) &= \int_{\Sigma \setminus B(\underline{z}, \epsilon)} \left( E_1(\underline{y} - \underline{z})(\underline{\nu}_1(\underline{y})(f(\underline{y}) - f(\underline{z})) + i(\tilde{f}(\underline{y}) - \tilde{f}(\underline{z}))\underline{\nu}_2(\underline{y})) \right. \\ & \quad \left. + ((f(\underline{y}) - f(\underline{z}))\underline{\nu}_2(\underline{y}) - i\underline{\nu}_1(\underline{y})(\tilde{f}(\underline{y}) - \tilde{f}(\underline{z})))E_2(\underline{y} - \underline{z}) \right) d\mathcal{H}^{2n-1}(\underline{y}). \end{aligned}$$

Note that for any  $f \in S^{isot}(\Sigma)$ , the isotonic singular integral operator  $\mathbb{S}_\Sigma^{isot} f$  exists for all  $\underline{z} \in \Sigma$  and it defines a continuous function on  $\Sigma$ .

It easily follows that

$$\mathbb{S}_\Sigma(fI) = (\mathbb{S}_\Sigma^{isot} f)I. \quad (5.6)$$

**Lemma 5.1** Let  $f$  be a  $\mathbb{C}_n$ -valued continuous function on  $\Sigma$ ,  $\underline{z} \in \Sigma$  and  $\epsilon > 0$ .

(i) If  $\underline{x} \in \Omega^+$  is such that  $|\underline{x} - \underline{z}| = \epsilon/2$ , then we have that

$$\begin{aligned} & |C_{\Sigma}^{isot} f(\underline{x}) - S_{\Sigma, \epsilon}^{isot} f(\underline{z}) - f(\underline{z})| \\ & \leq C \left( \frac{\theta_{\underline{z}}(\epsilon)}{(\text{dist}(\underline{x}, \Sigma))^{2n-1}} \omega_f(\epsilon) + \epsilon \int_{\epsilon}^d \frac{\omega_f(\tau)}{\tau^{2n}} d\theta_{\underline{z}}(\tau) \right). \end{aligned}$$

(ii) If  $\underline{x} \in \Omega^-$  is such that  $|\underline{x} - \underline{z}| = \epsilon/2$ , then we have that

$$\begin{aligned} & |C_{\Sigma}^{isot} f(\underline{x}) - S_{\Sigma, \epsilon}^{isot} f(\underline{z})| \\ & \leq C \left( \frac{\theta_{\underline{z}}(\epsilon)}{(\text{dist}(\underline{x}, \Sigma))^{2n-1}} \omega_f(\epsilon) + \epsilon \int_{\epsilon}^d \frac{\omega_f(\tau)}{\tau^{2n}} d\theta_{\underline{z}}(\tau) \right). \end{aligned}$$

*Proof.* Let  $\underline{x} \in \Omega^+$  with  $|\underline{x} - \underline{z}| = \epsilon/2$ . Then

$$\begin{aligned} & C_{\Sigma}^{isot} f(\underline{x}) - S_{\Sigma, \epsilon}^{isot} f(\underline{z}) - f(\underline{z}) \\ & = \int_{\Sigma \cap B(\underline{z}, \epsilon)} \left( E_1(\underline{y} - \underline{x}) (\nu_1(\underline{y}) (f(\underline{y}) - f(\underline{z})) + i(\tilde{f}(\underline{y}) - \tilde{f}(\underline{z})) \nu_2(\underline{y})) \right. \\ & \quad \left. + ((f(\underline{y}) - f(\underline{z})) \nu_2(\underline{y}) - i \nu_1(\underline{y}) (\tilde{f}(\underline{y}) - \tilde{f}(\underline{z}))) E_2(\underline{y} - \underline{x}) \right) d\mathcal{H}^{2n-1}(\underline{y}) \\ & \quad + \int_{\Sigma \setminus B(\underline{z}, \epsilon)} \left( (E_1(\underline{y} - \underline{x}) - E_1(\underline{y} - \underline{z})) (\nu_1(\underline{y}) (f(\underline{y}) - f(\underline{z})) + i(\tilde{f}(\underline{y}) - \tilde{f}(\underline{z})) \nu_2(\underline{y})) \right. \\ & \quad \left. + ((f(\underline{y}) - f(\underline{z})) \nu_2(\underline{y}) - i \nu_1(\underline{y}) (\tilde{f}(\underline{y}) - \tilde{f}(\underline{z}))) \right. \\ & \quad \left. \times (E_2(\underline{y} - \underline{x}) - E_2(\underline{y} - \underline{z})) \right) d\mathcal{H}^{2n-1}(\underline{y}). \end{aligned}$$

Let us denote by  $I_1$  and  $I_2$  the integrals on the right-hand side of the previous equality.

For  $I_1$  we obtain

$$\begin{aligned}
 |I_1| &\leq C \int_{\Sigma \cap B(\underline{z}, \epsilon)} \frac{|f(\underline{y}) - f(\underline{z})|}{|\underline{y} - \underline{x}|^{2n-1}} d\mathcal{H}^{2n-1}(\underline{y}) \\
 &\leq C \int_{\Sigma \cap B(\underline{z}, \epsilon)} \frac{\omega_f(|\underline{y} - \underline{z}|)}{|\underline{y} - \underline{x}|^{2n-1}} d\mathcal{H}^{2n-1}(\underline{y}) \\
 &\leq C \frac{\omega_f(\epsilon)}{(\text{dist}(\underline{x}, \Sigma))^{2n-1}} \int_{\Sigma \cap B(\underline{z}, \epsilon)} d\mathcal{H}^{2n-1}(\underline{y}) \\
 &= C \frac{\theta_{\underline{z}}(\epsilon)}{(\text{dist}(\underline{x}, \Sigma))^{2n-1}} \omega_f(\epsilon).
 \end{aligned}$$

To estimate  $I_2$ , we note that

$$|E_k(\underline{y} - \underline{x}) - E_k(\underline{y} - \underline{z})| \leq C|\underline{x} - \underline{z}| \sum_{j=0}^{2n-1} \frac{1}{|\underline{y} - \underline{x}|^{2n-j} |\underline{y} - \underline{z}|^j}, \quad k = 1, 2.$$

Now for  $\underline{y} \in \Sigma \setminus B(\underline{z}, \epsilon)$  we have that

$$\epsilon \leq |\underline{y} - \underline{z}| \leq |\underline{y} - \underline{x}| + |\underline{x} - \underline{z}| = |\underline{y} - \underline{x}| + \epsilon/2,$$

and therefore  $|\underline{y} - \underline{z}| \leq 2|\underline{y} - \underline{x}|$ . It follows that

$$|E_k(\underline{y} - \underline{x}) - E_k(\underline{y} - \underline{z})| \leq C \frac{|\underline{x} - \underline{z}|}{|\underline{y} - \underline{z}|^{2n}}, \quad \underline{y} \in \Sigma \setminus B(\underline{z}, \epsilon), \quad k = 1, 2.$$

We thus get

$$|I_2| \leq C|\underline{x} - \underline{z}| \int_{\Sigma \setminus B(\underline{z}, \epsilon)} \frac{\omega_f(|\underline{y} - \underline{z}|)}{|\underline{y} - \underline{z}|^{2n}} d\mathcal{H}^{2n-1}(\underline{y}) \leq C\epsilon \int_{\epsilon}^d \frac{\omega_f(\tau)}{\tau^{2n}} d\theta_{\underline{z}}(\tau),$$

which completes the proof of (i). In a similar way we can prove (ii).  $\square$

We will now derive the Plemelj-Sokhotski formulae for the isotonic Cauchy type integral. These formulae can be deduced from Lemma 5.1, but we will give a simpler proof using (5.5), (5.6) and Theorem 1.5. Nevertheless, Lemma 5.1 will prove extremely useful in the last chapter.

**Theorem 5.3** *Suppose that  $\Sigma$  is a rectifiable and AD-regular surface and let  $f$  be a  $\mathbb{C}_n$ -valued continuous function on  $\Sigma$ . Then  $C_\Sigma^{isot} f$  has continuous limit values on  $\Sigma$  if and only if  $f \in S^{isot}(\Sigma)$ . Moreover, the Plemelj-Sokhotski formulae for  $C_\Sigma^{isot} f$  hold:*

$$\lim_{\Omega^\pm \ni \underline{x} \rightarrow \underline{z}} C_\Sigma^{isot} f(\underline{x}) = \frac{1}{2} (S_\Sigma^{isot} f(\underline{z}) \pm f(\underline{z})), \quad \underline{z} \in \Sigma. \quad (5.7)$$

*Proof.* If  $C_\Sigma^{isot} f$  has continuous limit values on  $\Sigma$ , so does  $C_\Sigma(fI)$ , which follows from (5.5). The remark following Theorem 1.5 shows that  $fI \in S(\Sigma)$  and hence  $f \in S^{isot}(\Sigma)$ .

Now let us suppose that  $f \in S^{isot}(\Sigma)$ , then  $fI \in S(\Sigma)$ . Then, Theorem 1.5 implies that  $C_\Sigma(fI)$  has continuous limit values on  $\Sigma$ . From (5.5) we deduce that  $C_\Sigma^{isot} f$  also has continuous limit values on  $\Sigma$ .

Finally, for  $\underline{z} \in \Sigma$  we get

$$\begin{aligned} \lim_{\Omega^\pm \ni \underline{x} \rightarrow \underline{z}} (C_\Sigma^{isot} f(\underline{x}))I &= \lim_{\Omega^\pm \ni \underline{x} \rightarrow \underline{z}} C_\Sigma(f(\underline{x})I) \\ &= \frac{1}{2} (S_\Sigma(f(\underline{z})I) \pm f(\underline{z})I) \\ &= \frac{1}{2} (S_\Sigma^{isot} f(\underline{z}) \pm f(\underline{z})) I, \end{aligned}$$

where we have used Theorem 1.5 and (5.6).

Using (5.2) we obtain

$$\lim_{\Omega^\pm \ni \underline{x} \rightarrow \underline{z}} C_\Sigma^{isot} f(\underline{x}) = \frac{1}{2} (S_\Sigma^{isot} f(\underline{z}) \pm f(\underline{z})),$$

which completes the proof.  $\square$

Before continuing a few remarks need to be made. First, note that the rectifiability of  $\Sigma$  is only used to prove the necessity. Second, if  $f \in H_\varphi(\Sigma)$ , then clearly  $fI \in H_\varphi(\Sigma)$ . Therefore, if  $\Sigma$  is an AD-regular surface and  $\varphi$  is a regular majorant, then  $H_\varphi(\Sigma) \subset S^{isot}(\Sigma)$ . Finally, also for surfaces

of finite  $(2n - 1)$ -dimensional Hausdorff measure it is possible to prove the validity of the Plemelj-Sokhotski formulae (5.7). The following result may be proved in much the same way as Theorem 5.3 using Davydov's theorem for the Cauchy type integral provided in [5].

**Theorem 5.4** *Suppose that  $f$  is a  $\mathbb{C}_n$ -valued continuous function on  $\Sigma$  and that the principal value integral*

$$\lim_{\epsilon \rightarrow 0^+} \int_{\Sigma \setminus B(\underline{z}, \epsilon)} \frac{|f(\underline{y}) - f(\underline{z})|}{|\underline{y} - \underline{z}|^{2n-1}} d\mathcal{H}^{2n-1}(\underline{y})$$

*exists uniformly with respect to  $\underline{z}$  on  $\Sigma$ . Then  $\mathbb{C}_\Sigma^{isot} f$  has continuous limit values on  $\Sigma$  given by (5.7).*

The question under which conditions a continuous function  $f$  on the boundary  $\Sigma$  has an isotonic extension to  $\Omega^+$ , is easily answered in the following theorem.

**Theorem 5.5** *Let  $\Sigma$  be an AD-regular surface and let  $f \in S^{isot}(\Sigma)$ . Then  $f$  has an isotonic extension to  $\Omega^+$  if and only if  $\mathbb{S}_\Sigma^{isot} f = f$  on  $\Sigma$ .*

*Proof.* Let  $F$  be an isotonic extension of  $f$  to  $\Omega^+$ . By Theorem 5.2, we have that

$$F(\underline{x}) = \mathbb{C}_\Sigma^{isot} f(\underline{x}), \quad \underline{x} \in \Omega^+.$$

By Theorem 5.3, it then follows that

$$f(\underline{z}) = \frac{1}{2} (\mathbb{S}_\Sigma^{isot} f(\underline{z}) + f(\underline{z})), \quad \underline{z} \in \Sigma,$$

and hence  $\mathbb{S}_\Sigma^{isot} f = f$  on  $\Sigma$ . Conversely, if  $\mathbb{S}_\Sigma^{isot} f = f$  on  $\Sigma$ , then it follows from (5.7) that

$$F(\underline{x}) = \begin{cases} \mathbb{C}_\Sigma^{isot} f(\underline{x}) & \text{for } \underline{x} \in \Omega^+, \\ f(\underline{x}) & \text{for } \underline{x} \in \Sigma, \end{cases}$$

is an isotonic extension of  $f$  to  $\Omega^+$ . □

We end this section with two results concerning the isotonic singular integral operator.

**Theorem 5.6** *Let  $\Sigma$  be an AD-regular surface. Then the isotonic singular integral operator  $S_\Sigma^{isot}$  is an involution on  $S^{isot}(\Sigma)$ , i.e.  $(S_\Sigma^{isot})^2 f = f$  for all  $f \in S^{isot}(\Sigma)$ .*

*Proof.* Using Theorem 1.6 and (5.6), we have

$$fI = S_\Sigma^2(fI) = \left( (S_\Sigma^{isot})^2 f \right) I.$$

From (5.2) we obtain  $(S_\Sigma^{isot})^2 f = f$ , which is the desired conclusion.  $\square$

**Theorem 5.7** *Assume that  $\Sigma$  is an AD-regular surface and let  $\varphi$  be a regular majorant. Then  $S_\Sigma^{isot}$  is a bounded operator mapping  $H_\varphi(\Sigma)$  into itself.*

*Proof.* The proof easily follows using Theorem 1.7 and (5.6).  $\square$

### 5.3 Bochner-Martinelli type integrals

Let  $\Sigma$  be an AD-regular surface and let  $\varphi$  be a regular majorant. If  $F_k \in H_\varphi(\Sigma)$  is a  $\mathbb{C}_n^{(k)}$ -valued function, then the isotonic Cauchy type integral of  $F_k$  splits into

$$C_\Sigma^{isot} F_k(\underline{x}) = [C_\Sigma^{isot} F_k(\underline{x})]_{k-2} + [C_\Sigma^{isot} F_k(\underline{x})]_k + [C_\Sigma^{isot} F_k(\underline{x})]_{k+2}, \quad \underline{x} \in \mathbb{R}^{2n} \setminus \Sigma,$$

where  $[C_\Sigma^{isot} F_k]_{k-2}$ ,  $[C_\Sigma^{isot} F_k]_k$  and  $[C_\Sigma^{isot} F_k]_{k+2}$  are Bochner-Martinelli type integrals given by

$$\begin{aligned} [C_\Sigma^{isot} F_k(\underline{x})]_{k-2} &= \int_\Sigma \left( E_1(\underline{y} - \underline{x}) \bullet ((\underline{\nu}_1(\underline{y}) - i\underline{\nu}_2(\underline{y})) \bullet F_k(\underline{y})) \right. \\ &\quad \left. + (F_k(\underline{y}) \bullet (\underline{\nu}_2(\underline{y}) + i\underline{\nu}_1(\underline{y}))) \bullet E_2(\underline{y} - \underline{x}) \right) d\mathcal{H}^{2n-1}(\underline{y}), \end{aligned}$$

$$[C_\Sigma^{isot} F_k(\underline{x})]_k = \int_\Sigma \left( E_1(\underline{y} - \underline{x}) \wedge ((\underline{\nu}_1(\underline{y}) - i\underline{\nu}_2(\underline{y})) \bullet F_k(\underline{y})) \right)$$



$$\begin{aligned}
& + E_1(\underline{y} - \underline{x}) \bullet ((\nu_1(\underline{y}) + i\nu_2(\underline{y})) \wedge F_k(\underline{y})) \\
& + (F_k(\underline{y}) \bullet (\nu_2(\underline{y}) + i\nu_1(\underline{y}))) \wedge E_2(\underline{y} - \underline{x}) \\
& + (F_k(\underline{y}) \wedge (\nu_2(\underline{y}) - i\nu_1(\underline{y}))) \bullet E_2(\underline{y} - \underline{x}) \Big) d\mathcal{H}^{2n-1}(\underline{y}), \\
[\mathbf{C}_\Sigma^{isot} F_k(\underline{x})]_{k+2} &= \int_\Sigma \Big( E_1(\underline{y} - \underline{x}) \wedge ((\nu_1(\underline{y}) + i\nu_2(\underline{y})) \wedge F_k(\underline{y})) \\
& + (F_k(\underline{y}) \wedge (\nu_2(\underline{y}) - i\nu_1(\underline{y}))) \wedge E_2(\underline{y} - \underline{x}) \Big) d\mathcal{H}^{2n-1}(\underline{y}).
\end{aligned}$$

In particular, if  $k = 0$  (i.e. for  $\mathbb{C}$ -valued functions), then  $[\mathbf{C}_\Sigma^{isot} F_k]_k$  is the classical Bochner-Martinelli integral which is an important object in the theory of functions of several complex variables (see [75]).

In a similar way we see that for  $\underline{z} \in \Sigma$ ,

$$\mathbf{S}_\Sigma^{isot} F_k(\underline{z}) = [\mathbf{S}_\Sigma^{isot} F_k(\underline{z})]_{k-2} + [\mathbf{S}_\Sigma^{isot} F_k(\underline{z})]_k + [\mathbf{S}_\Sigma^{isot} F_k(\underline{z})]_{k+2},$$

where

$$\begin{aligned}
[\mathbf{S}_\Sigma^{isot} F_k(\underline{z})]_{k-2} &= 2 \int_\Sigma \Big( E_1(\underline{y} - \underline{z}) \bullet ((\nu_1(\underline{y}) - i\nu_2(\underline{y})) \bullet (F_k(\underline{y}) - F_k(\underline{z}))) \\
& + ((F_k(\underline{y}) - F_k(\underline{z})) \bullet (\nu_2(\underline{y}) + i\nu_1(\underline{y}))) \bullet E_2(\underline{y} - \underline{z}) \Big) d\mathcal{H}^{2n-1}(\underline{y}), \\
[\mathbf{S}_\Sigma^{isot} F_k(\underline{z})]_k &= 2 \int_\Sigma \Big( E_1(\underline{y} - \underline{z}) \wedge ((\nu_1(\underline{y}) - i\nu_2(\underline{y})) \bullet (F_k(\underline{y}) - F_k(\underline{z}))) \\
& + E_1(\underline{y} - \underline{z}) \bullet ((\nu_1(\underline{y}) + i\nu_2(\underline{y})) \wedge (F_k(\underline{y}) - F_k(\underline{z}))) \\
& + ((F_k(\underline{y}) - F_k(\underline{z})) \bullet (\nu_2(\underline{y}) + i\nu_1(\underline{y}))) \wedge E_2(\underline{y} - \underline{z}) \\
& + ((F_k(\underline{y}) - F_k(\underline{z})) \wedge (\nu_2(\underline{y}) - i\nu_1(\underline{y}))) \bullet E_2(\underline{y} - \underline{z}) \Big) d\mathcal{H}^{2n-1}(\underline{y}) \\
& + F_k(\underline{z}), \\
[\mathbf{S}_\Sigma^{isot} F_k(\underline{z})]_{k+2} &= 2 \int_\Sigma \Big( E_1(\underline{y} - \underline{z}) \wedge ((\nu_1(\underline{y}) + i\nu_2(\underline{y})) \wedge (F_k(\underline{y}) - F_k(\underline{z}))) \\
& + ((F_k(\underline{y}) - F_k(\underline{z})) \wedge (\nu_2(\underline{y}) - i\nu_1(\underline{y}))) \wedge E_2(\underline{y} - \underline{z}) \Big) d\mathcal{H}^{2n-1}(\underline{y}).
\end{aligned}$$

In view of the above decompositions, we thus obtain from Theorem 5.3:

**Theorem 5.8** *Let  $\Sigma$  be an AD-regular surface and let  $\varphi$  be a regular majorant. If  $F_k \in H_\varphi(\Sigma)$  is a  $\mathbb{C}_n^{(k)}$ -valued function, then  $[\mathbf{C}_\Sigma^{isot} F_k]_{k-2}$ ,  $[\mathbf{C}_\Sigma^{isot} F_k]_k$  and  $[\mathbf{C}_\Sigma^{isot} F_k]_{k+2}$  have continuous limit values on  $\Sigma$  given by*

- (i)  $\lim_{\Omega^\pm \ni \underline{x} \rightarrow \underline{z}} [\mathbf{C}_\Sigma^{isot} F_k(\underline{x})]_{k-2} = \frac{1}{2} [\mathbf{S}_\Sigma^{isot} F_k(\underline{z})]_{k-2},$
- (ii)  $\lim_{\Omega^\pm \ni \underline{x} \rightarrow \underline{z}} [\mathbf{C}_\Sigma^{isot} F_k(\underline{x})]_k = \frac{1}{2} ([\mathbf{S}_\Sigma^{isot} F_k(\underline{z})]_k \pm F_k(\underline{z})),$
- (iii)  $\lim_{\Omega^\pm \ni \underline{x} \rightarrow \underline{z}} [\mathbf{C}_\Sigma^{isot} F_k(\underline{x})]_{k+2} = \frac{1}{2} [\mathbf{S}_\Sigma^{isot} F_k(\underline{z})]_{k+2}.$

## Chapter 6

# Holomorphic and biregular extension theorems

The results of the previous chapter enable us to study the question under which conditions a  $\mathbb{C}$ -valued (resp.  $\mathbb{R}_{0,n}$ -valued) function defined on the boundary  $\Sigma$  has a holomorphic (resp. biregular) extension to  $\Omega^+$  (see [11, 12, 13]).

### 6.1 Holomorphic functions

Let  $m = 2n$ . We shall here identify  $\mathbb{R}^{2n}$  with  $\mathbb{C}^n$  by associating to any element  $(x_1, \dots, x_{2n}) \in \mathbb{R}^{2n}$  the complex vector  $(z_1, \dots, z_n) \in \mathbb{C}^n$  with  $z_j = x_j + ix_{n+j}$ ,  $j = 1, \dots, n$ .

The theory of several complex variables is a natural extension of classical complex analysis to the multivariable setting. For a detailed treatment we refer the reader to e.g. [63, 70, 74, 98].

**Definition 6.1** *A continuous function  $f : \Omega \rightarrow \mathbb{C}$  on an open set  $\Omega$  in  $\mathbb{R}^{2n}$  is said to be holomorphic in  $\Omega$  if  $f$  is holomorphic in each variable*

$z_j$  ( $j = 1, \dots, n$ ) separately, i.e. if it satisfies in  $\Omega$  the Cauchy-Riemann equations

$$(\partial_{x_j} + i\partial_{x_{n+j}})f = 0, \quad j = 1, \dots, n.$$

Many basic results of classical one variable complex analysis generalize in a natural way to several variables. However, also new and surprising phenomena emerge, an example of which is given in the following lemma (see [99]).

**Lemma 6.1** *Suppose that  $K$  is a compact subset of  $\mathbb{R}^{2n}$ ,  $n \geq 2$ , such that  $\mathbb{R}^{2n} \setminus K$  is connected. If  $f$  is holomorphic and bounded in  $\mathbb{R}^{2n} \setminus K$ , then  $f$  is a constant.*

*Proof.* By Hartogs' theorem, we have that  $f$  may be uniquely extended to a holomorphic function in  $\mathbb{R}^{2n}$ . Clearly, this extension is bounded in  $\mathbb{R}^{2n}$  and therefore is a constant by Liouville's theorem.  $\square$

## 6.2 Holomorphic extension for Hölder continuous functions

Let  $f$  be a  $\mathbb{C}$ -valued continuous function on  $\Sigma$ . The Bochner-Martinelli integral (see [75] and the references given there) is defined by

$$M_1 f(\underline{x}) = - \int_{\Sigma} \langle E_1(\underline{y} - \underline{x}) - iE_2(\underline{y} - \underline{x}), \underline{\nu}_1(\underline{y}) + i\underline{\nu}_2(\underline{y}) \rangle f(\underline{y}) d\mathcal{H}^{2n-1}(\underline{y}),$$

$$\underline{x} \in \mathbb{R}^{2n} \setminus \Sigma.$$

Aronov and Kytmanov provided in [16] (see also [75, 76]) the following characterizations:

*If  $\Sigma$  is a smooth surface and  $f$  is a continuously differentiable function in  $\Sigma$ , then a necessary and sufficient condition for  $f$  to have a holomorphic extension to  $\Omega^+$  is that  $M_1 f(\underline{x}) = 0$  for  $\underline{x} \in \Omega^-$ .*

If  $\Sigma$  is a piecewise smooth surface and  $F$  is a continuously differentiable function in  $\overline{\Omega^+}$ , then  $F$  is holomorphic in  $\Omega^+$  if and only if  $F(\underline{x}) = M_1 F(\underline{x})$  for  $\underline{x} \in \Omega^+$ .

In this section we give alternative characterizations using the results of the previous chapter. We will also re-establish the results of Aronov and Kytmanov for the particular case  $n = 2$  using our techniques.

We have already seen that

$$C_{\Sigma}^{isot} f(\underline{x}) = M_1 f(\underline{x}) + M_2 f(\underline{x}), \quad \underline{x} \in \mathbb{R}^{2n} \setminus \Sigma, \quad (6.1)$$

where  $M_2$  is a bivector-valued integral operator given by

$$M_2 f(\underline{x}) = \int_{\Sigma} (E_1(\underline{y} - \underline{x}) + iE_2(\underline{y} - \underline{x})) \wedge (\underline{\nu}_1(\underline{y}) + i\underline{\nu}_2(\underline{y})) f(\underline{y}) d\mathcal{H}^{2n-1}(\underline{y}).$$

From (6.1) it may be concluded that  $M_1$  and  $M_2$  are harmonic in  $\mathbb{R}^{2n} \setminus \Sigma$  and that  $M_1 f(\infty) = M_2 f(\infty) = 0$ .

We now assume that  $\Sigma$  is an AD-regular surface and that  $f$  is a  $\mathbb{C}$ -valued function which belongs to  $H_{\varphi}(\Sigma)$ , where  $\varphi$  is a regular majorant. We thus have

$$S_{\Sigma}^{isot} f(\underline{z}) = N_1 f(\underline{z}) + N_2 f(\underline{z}), \quad \underline{z} \in \Sigma, \quad (6.2)$$

where  $N_1$  and  $N_2$  are the singular versions of  $M_1$  and  $M_2$  respectively, given by

$$N_1 f(\underline{z}) = 2 \lim_{\epsilon \rightarrow 0+} N_{1,\epsilon} f(\underline{z}) + f(\underline{z}),$$

with

$$N_{1,\epsilon} f(\underline{z}) = - \int_{\Sigma \setminus B(\underline{z}, \epsilon)} \langle E_1(\underline{y} - \underline{z}) - iE_2(\underline{y} - \underline{z}), \underline{\nu}_1(\underline{y}) + i\underline{\nu}_2(\underline{y}) \rangle (f(\underline{y}) - f(\underline{z})) d\mathcal{H}^{2n-1}(\underline{y})$$

and

$$N_2 f(\underline{z}) = 2 \lim_{\epsilon \rightarrow 0+} N_{2,\epsilon} f(\underline{z}),$$

with

$$N_{2,\epsilon}f(\underline{z}) = \int_{\Sigma \setminus B(\underline{z}, \epsilon)} (E_1(\underline{y} - \underline{z}) + iE_2(\underline{y} - \underline{z})) \wedge (\underline{\nu}_1(\underline{y}) + i\underline{\nu}_2(\underline{y})) \\ \times (f(\underline{y}) - f(\underline{z})) d\mathcal{H}^{2n-1}(\underline{y}).$$

It is natural to ask whether the Bochner-Martinelli singular integral operator  $N_1f$  is an involution. The following theorem provides an answer to this question (see also [100, 121]).

**Theorem 6.1** *Assume that  $\Sigma$  is an AD-regular surface and let  $\varphi$  be a regular majorant. Then  $N_1$  and  $N_2$  are bounded operators mapping  $H_\varphi(\Sigma)$  into itself. Moreover, the formulae*

$$\begin{aligned} N_1^2f + [N_2^2f]_0 &= f, \\ N_1N_2f + N_2N_1f + [N_2^2f]_2 &= 0, \\ [N_2^2f]_4 &= 0, \end{aligned}$$

hold for all  $f \in H_\varphi(\Sigma)$ .

*Proof.* The proof easily follows using (6.2) as well as Theorems 5.6 and 5.7.  $\square$

It is worth noting that the formulae above were obtained in [15] (see also [106]) for the case  $n = 2$ .

As an application of Theorem 5.8 we obtain that  $M_1$  and  $M_2$  have continuous limit values on  $\Sigma$  given by the formulae

$$M_1^\pm f(\underline{z}) = \lim_{\Omega^\pm \ni \underline{x} \rightarrow \underline{z}} M_1f(\underline{x}) = \frac{1}{2}(N_1f(\underline{z}) \pm f(\underline{z})), \quad \underline{z} \in \Sigma, \quad (6.3)$$

$$M_2^\pm f(\underline{z}) = \lim_{\Omega^\pm \ni \underline{x} \rightarrow \underline{z}} M_2f(\underline{x}) = \frac{1}{2}N_2f(\underline{z}), \quad \underline{z} \in \Sigma. \quad (6.4)$$

Note that (6.3) are the Plemelj-Sokhotski formulae for the Bochner-Martinelli integral (see e.g. [15, 32, 59, 60, 75, 79, 82]).

It is not difficult to show that a necessary and sufficient condition for  $f$  to have a holomorphic extension to  $\Omega^+$  is that

$$N_1 f = f \text{ on } \Sigma, \quad (6.5)$$

$$M_2 f = 0 \text{ in } \Omega^+. \quad (6.6)$$

Indeed, let us suppose that  $F$  is a holomorphic extension of  $f$  to  $\Omega^+$ . Corollary 5.1 now yields (6.6) and

$$F(\underline{x}) = M_1 f(\underline{x}), \quad \underline{x} \in \Omega^+.$$

From (6.3) it follows that

$$f(\underline{z}) = \frac{1}{2}(N_1 f(\underline{z}) + f(\underline{z})), \quad \underline{z} \in \Sigma,$$

and hence  $N_1 f = f$  on  $\Sigma$ . Conversely, if (6.5) and (6.6) hold, then from (6.5) we can deduce that  $M_1 f$  is a harmonic extension of  $f$  to  $\Omega^+$ , while (6.6) implies that  $M_1 f$  is isotonic in  $\Omega^+$  and hence holomorphic in  $\Omega^+$ .

What is more, in the next theorem we will show that condition (6.5) is redundant.

**Theorem 6.2** *Let  $\Sigma$  be an AD-regular surface and let  $\varphi$  be a regular majorant. Suppose that  $f$  is a  $\mathbb{C}$ -valued function which belongs to  $H_\varphi(\Sigma)$ . Then the following statements are equivalent:*

- (i)  $f$  has a holomorphic extension to  $\Omega^+$ ;
- (ii)  $M_2 f = 0$  in  $\Omega^+$ ;
- (iii)  $N_2 f = 0$  on  $\Sigma$ ;
- (iv)  $M_2 f = 0$  in  $\Omega^-$ .

*Proof.* (i)  $\Rightarrow$  (ii): This easily follows from statement (ii) of Corollary 5.1.

(ii)  $\Leftrightarrow$  (iii)  $\Leftrightarrow$  (iv): If  $M_2f(\underline{x}) = 0$  for all  $\underline{x} \in \Omega^+$ , then from (6.4) we obtain  $N_2f = 0$  on  $\Sigma$ , and hence  $M_2^-f = 0$  on  $\Sigma$ . Since  $M_2f$  is harmonic in  $\Omega^-$  and vanishes at infinity, it follows that  $M_2f(\underline{x}) = 0$  for all  $\underline{x} \in \Omega^-$ . In the same way we can show that (iv)  $\Rightarrow$  (ii).

(iv)  $\Rightarrow$  (i): Now assume that  $M_2f(\underline{x}) = 0$  for all  $\underline{x} \in \Omega^-$ . From (6.1) we see that  $M_1f$  is isotonic in  $\Omega^-$  and hence holomorphic in  $\Omega^-$ . Lemma 6.1 now shows that  $M_1f = 0$  in  $\Omega^-$ . This gives  $N_1f = f$  on  $\Sigma$ , which follows from (6.3), and consequently  $M_1^+f = f$  on  $\Sigma$ . It only remains to show that  $M_1f$  is holomorphic in  $\Omega^+$ . This follows using (6.1) and the fact that (iv)  $\Rightarrow$  (ii).  $\square$

We have already proved that  $M_2f = 0$  in  $\Omega^-$  implies  $M_1f = 0$  in  $\Omega^-$ . Using our techniques, we will prove how the inverse assertion may be deduced for the case  $n = 2$ . Indeed, if  $M_1f = 0$  in  $\Omega^-$ , then we have that  $M_2f$  is isotonic in  $\Omega^-$ . Since  $n = 2$ , it follows that  $M_2f = ge_1e_2$  for some  $\mathbb{C}$ -valued function  $g$ . It is easy to check that a function of this form is isotonic if and only if the function  $g$  is antiholomorphic, i.e.  $\bar{g}$  is holomorphic. Therefore  $M_2f = 0$  in  $\Omega^-$  by Lemma 6.1. We have thus obtained an alternative proof of the first result of Aronov and Kytmanov for  $n = 2$ .

An easy consequence of Theorem 6.2 is the following corollary.

**Corollary 6.1** *Let  $\Sigma$  be an AD-regular surface and let  $\varphi$  be a regular majorant. Suppose that  $F$  is a  $\mathbb{C}$ -valued continuous function on  $\overline{\Omega^+}$  such that  $f = F|_\Sigma \in H_\varphi(\Sigma)$ . A necessary and sufficient condition for  $F$  to be holomorphic in  $\Omega^+$  is that  $F$  is harmonic in  $\Omega^+$  and that  $M_2f(\underline{x}) = 0$  for all  $\underline{x} \in \Omega^+$ .*

*Proof.* If  $F$  is holomorphic in  $\Omega^+$ , then obviously  $F$  is harmonic in  $\Omega^+$  and  $M_2f(\underline{x}) = 0$  in  $\Omega^+$  by Corollary 5.1. Now, if  $M_2f(\underline{x}) = 0$  for all  $\underline{x} \in \Omega^+$ , then by Theorem 6.2 the function

$$G(\underline{x}) = \begin{cases} M_1f(\underline{x}) & \text{for } \underline{x} \in \Omega^+, \\ f(\underline{x}) & \text{for } \underline{x} \in \Sigma, \end{cases}$$



is a holomorphic extension of  $f$  to  $\Omega^+$ . As  $F - G$  is harmonic in  $\Omega^+$  and  $(F - G)|_\Sigma = 0$  we have  $F(\underline{x}) = M_1 f(\underline{x})$  for all  $\underline{x} \in \Omega^+$ , which follows from the maximum principle for harmonic functions.  $\square$

Note that if  $F = M_1 f$  in  $\Omega^+$ , then clearly  $F$  is harmonic in  $\Omega^+$ . Using (6.3) we also obtain  $N_1 f = f$  on  $\Sigma$ , and hence  $M_1^- f = 0$  on  $\Sigma$ . The maximum principle for harmonic functions now yields  $M_1 f = 0$  in  $\Omega^-$ . This completes the proof of the second result of Aronov and Kytmanov for  $n = 2$ .

It is also worth remarking that our assumptions on  $f$  and  $\Sigma$  are less restrictive than Aronov-Kytmanov's assumptions.

### 6.3 Holomorphic extension for continuous functions

In Theorem 6.2 we have assumed that  $f$  belongs to some space of generalized Hölder continuous functions  $H_\varphi(\Sigma)$ , with  $\varphi$  a regular majorant. An obvious question to ask is whether the assertion of Theorem 6.2 continues to hold for merely continuous functions on  $\Sigma$ .

It is the final aim of this section to answer that question, but first we will prove that the Plemelj-Sokhotski formulae (6.3) and (6.4) are still valid for a subclass of continuous functions wider than  $H_\varphi(\Sigma)$ .

**Theorem 6.3** *Let  $\Sigma$  be an AD-regular surface and let  $f$  be a  $\mathbb{C}$ -valued continuous function on  $\Sigma$ . If the integrals*

$$\int_{\Sigma \cap B(\underline{z}, \epsilon)} \langle E_1(\underline{y} - \underline{z}) - iE_2(\underline{y} - \underline{z}), \nu_1(\underline{y}) + i\nu_2(\underline{y}) \rangle (f(\underline{y}) - f(\underline{z})) d\mathcal{H}^{2n-1}(\underline{y})$$

*converge uniformly to zero for  $\underline{z} \in \Sigma$  as  $\epsilon \rightarrow 0$ , then the Bochner-Martinelli integral  $M_1 f$  has continuous limit values on  $\Sigma$  given by (6.3).*

*Proof.* We restrict ourselves to the proof of the statement for  $M_1^+ f$ , the proof of the one for  $M_1^- f$  being similar.

Let  $\underline{z}$  be a fixed point of  $\Sigma$  and let  $\underline{x} \in \Omega^+$ . If  $\underline{z}_{\underline{x}} \in \{\underline{y} \in \Sigma : |\underline{y} - \underline{x}| = \text{dist}(\underline{x}, \Sigma)\}$ , we have that

$$\begin{aligned} \left| M_1 f(\underline{x}) - \frac{1}{2} (N_1 f(\underline{z}) + f(\underline{z})) \right| &\leq |M_1 f(\underline{x}) - N_{1,\epsilon} f(\underline{z}_{\underline{x}}) - f(\underline{z}_{\underline{x}})| \\ &+ \left| \int_{\Sigma \cap B(\underline{z}_{\underline{x}}, \epsilon)} \langle E_1(\underline{y} - \underline{z}_{\underline{x}}) - iE_2(\underline{y} - \underline{z}_{\underline{x}}), \underline{\nu}_1(\underline{y}) + i\underline{\nu}_2(\underline{y}) \rangle \right. \\ &\quad \left. \times (f(\underline{y}) - f(\underline{z}_{\underline{x}})) d\mathcal{H}^{2n-1}(\underline{y}) \right| \\ &+ |N_1 f(\underline{z}_{\underline{x}}) - N_1 f(\underline{z})| + |f(\underline{z}_{\underline{x}}) - f(\underline{z})|, \end{aligned} \quad (6.7)$$

with  $\epsilon = \text{dist}(\underline{x}, \Sigma)$ .

By Lemma 5.1 and using the fact that  $\Sigma$  is an AD-regular surface we can deduce that

$$|C_{\Sigma}^{isot} f(\underline{x}) - S_{\Sigma, \epsilon}^{isot} f(\underline{z}_{\underline{x}}) - f(\underline{z}_{\underline{x}})| \leq C \left( \omega_f(\epsilon) + \epsilon \int_{\epsilon}^d \frac{\omega_f(\tau)}{\tau^2} d\tau \right).$$

From the above it follows that

$$|M_1 f(\underline{x}) - N_{1,\epsilon} f(\underline{z}_{\underline{x}}) - f(\underline{z}_{\underline{x}})| \leq C \left( \omega_f(\epsilon) + \epsilon \int_{\epsilon}^d \frac{\omega_f(\tau)}{\tau^2} d\tau \right).$$

By the assumptions on  $f$  and using the last inequality, it is easily seen that the right-hand side of (6.7) tends to zero as  $\underline{x} \rightarrow \underline{z}$ .  $\square$

We note that Theorem 6.3 was obtained by Gaziev for sufficiently smooth surfaces in [59, 60].

In a similar way we can prove the following.

**Theorem 6.4** *Let  $\Sigma$  be an AD-regular surface and let  $f$  be a  $\mathbb{C}$ -valued continuous function on  $\Sigma$ . If the integrals*

$$\int_{\Sigma \cap B(\underline{z}, \epsilon)} (E_1(\underline{y} - \underline{z}) + iE_2(\underline{y} - \underline{z})) \wedge (\underline{\nu}_1(\underline{y}) + i\underline{\nu}_2(\underline{y})) (f(\underline{y}) - f(\underline{z})) d\mathcal{H}^{2n-1}(\underline{y})$$

converge uniformly to zero for  $\underline{z} \in \Sigma$  as  $\epsilon \rightarrow 0$ , then  $M_2 f$  has continuous limit values on  $\Sigma$  given by (6.4).

The following results may be easily deduced from Lemma 5.1.

**Lemma 6.2** *Let  $f$  be a  $\mathbb{C}$ -valued continuous function on  $\Sigma$ ,  $\underline{z} \in \Sigma$  and  $\epsilon > 0$ .*

(i) *If  $\underline{x} \in \Omega^+$  is such that  $|\underline{x} - \underline{z}| = \epsilon/2$ , then we have that*

$$\begin{aligned} & |M_1 f(\underline{x}) - N_{1,\epsilon} f(\underline{z}) - f(\underline{z})| \\ & \leq C \left( \frac{\theta_{\underline{z}}(\epsilon)}{(\text{dist}(\underline{x}, \Sigma))^{2n-1}} \omega_f(\epsilon) + \epsilon \int_{\epsilon}^d \frac{\omega_f(\tau)}{\tau^{2n}} d\theta_{\underline{z}}(\tau) \right). \end{aligned}$$

(ii) *If  $\underline{x} \in \Omega^-$  is such that  $|\underline{x} - \underline{z}| = \epsilon/2$ , then we have that*

$$\begin{aligned} & |M_1 f(\underline{x}) - N_{1,\epsilon} f(\underline{z})| \\ & \leq C \left( \frac{\theta_{\underline{z}}(\epsilon)}{(\text{dist}(\underline{x}, \Sigma))^{2n-1}} \omega_f(\epsilon) + \epsilon \int_{\epsilon}^d \frac{\omega_f(\tau)}{\tau^{2n}} d\theta_{\underline{z}}(\tau) \right). \end{aligned}$$

**Lemma 6.3** *Let  $f$  be a  $\mathbb{C}$ -valued continuous function on  $\Sigma$ ,  $\underline{z} \in \Sigma$  and  $\epsilon > 0$ . Then we have that*

$$|M_2 f(\underline{x}) - N_{2,\epsilon} f(\underline{z})| \leq C \left( \frac{\theta_{\underline{z}}(\epsilon)}{(\text{dist}(\underline{x}, \Sigma))^{2n-1}} \omega_f(\epsilon) + \epsilon \int_{\epsilon}^d \frac{\omega_f(\tau)}{\tau^{2n}} d\theta_{\underline{z}}(\tau) \right),$$

where  $\underline{x} \in \Omega^{\pm}$  is such that  $|\underline{x} - \underline{z}| = \epsilon/2$ .

We remark that Lemma 6.2 have been previously proved by Gaziev for sufficiently smooth surfaces in [58].

In what follows, we will assume that  $\Omega^+$  is a Lipschitz domain, i.e. its boundary  $\Sigma$  is locally the graph of a Lipschitz continuous function (see e.g. [61]).

It is well-known that bounded Lipschitz domains satisfy the so-called uniform interior and exterior cone condition. That is, there exists constants  $\lambda > 0$ ,  $\delta > 0$  such that for every  $\underline{z} \in \Sigma$ , one of the two components of  $V(\underline{z}) \cap \{\underline{\zeta} : |\underline{\zeta} - \underline{z}| < \delta\}$  is completely contained in  $\Omega^+$  and the other is completely contained in  $\Omega^-$ , where

$$V(\underline{z}) = \{\underline{x} : |\underline{x} - \underline{z}| \leq (1 + \lambda) \text{dist}(\underline{x}, \Sigma)\}.$$

**Theorem 6.5** *Let  $\Omega^+$  be a Lipschitz domain. Suppose that  $f$  is a  $\mathbb{C}$ -valued continuous function on  $\Sigma$ . Then  $f$  has a holomorphic extension to  $\Omega^+$  if and only if  $M_2 f = 0$  in  $\Omega^+$ .*

*Proof.* The necessity is obvious. Thus, we show the sufficiency. Suppose that  $M_2 f = 0$  in  $\Omega^+$ . It follows that  $M_1 f$  is holomorphic in  $\Omega^+$ . Now if  $\underline{z} \in \Sigma$  and  $\epsilon > 0$ , then by Lemma 6.3 we get that

$$|N_{2,\epsilon} f(\underline{z})| \leq C \left( \frac{\theta_{\underline{z}}(\epsilon)}{(\text{dist}(\underline{x}, \Sigma))^{2n-1}} \omega_f(\epsilon) + \epsilon \int_{\epsilon}^d \frac{\omega_f(\tau)}{\tau^{2n}} d\theta_{\underline{z}}(\tau) \right),$$

where  $\underline{x} \in \Omega^+$  is such that  $|\underline{x} - \underline{z}| = \epsilon/2$ . When  $\underline{x}$  approaches  $\underline{z}$  non-tangentially inside the cone  $V(\underline{z})$ , the uniform interior and exterior cone condition implies that

$$\frac{\epsilon}{2} = |\underline{x} - \underline{z}| \leq (1 + \lambda) \text{dist}(\underline{x}, \Sigma).$$

Combining the above inequality with the AD-regularity of  $\Sigma$ , we obtain

$$|N_{2,\epsilon} f(\underline{z})| \leq C \left( \omega_f(\epsilon) + \epsilon \int_{\epsilon}^d \frac{\omega_f(\tau)}{\tau^2} d\tau \right).$$

Therefore  $N_{2,\epsilon} f(\underline{z})$  converges uniformly on  $\Sigma$  as  $\epsilon \rightarrow 0$ . Theorem 6.4 now shows that  $M_2 f$  has continuous limit values on  $\Sigma$  given by (6.4). This clearly forces  $M_2 f = 0$  in  $\Omega^-$  and hence  $M_1 f = 0$  in  $\Omega^-$ .

By Lemma 6.2 we thus get that

$$|N_{1,\epsilon} f(\underline{z})| \leq C \left( \frac{\theta_{\underline{z}}(\epsilon)}{(\text{dist}(\underline{x}, \Sigma))^{2n-1}} \omega_f(\epsilon) + \epsilon \int_{\epsilon}^d \frac{\omega_f(\tau)}{\tau^{2n}} d\theta_{\underline{z}}(\tau) \right),$$

where  $\underline{x} \in \Omega^-$  is such that  $|\underline{x} - \underline{z}| = \epsilon/2$ . In the same way we can see that  $N_{1,\epsilon}f(\underline{z})$  converges uniformly on  $\Sigma$  as  $\epsilon \rightarrow 0$ . Then, Theorem 6.3 implies that  $M_1f$  has continuous limit values on  $\Sigma$  given by (6.3). This gives  $N_1f = f$  on  $\Sigma$ , which completes the proof.  $\square$

We get the following corollary from this theorem just as we did from Theorem 6.2.

**Corollary 6.2** *Let  $\Omega^+$  be a Lipschitz domain. Suppose that  $F$  is a  $\mathbb{C}$ -valued continuous function on  $\overline{\Omega^+}$ . A necessary and sufficient condition for  $F$  to be holomorphic in  $\Omega^+$  is that  $F$  is harmonic in  $\Omega^+$  and that  $M_2F(\underline{x}) = 0$  for all  $\underline{x} \in \Omega^+$ .*

## 6.4 Biregular extension for Hölder continuous functions

The aim of this section is to generalize the results of Aronov and Kytmanov to the case of biregular functions.

First we recall the definition of a biregular function. Let  $m = 2n$  and consider the real Clifford algebra  $\mathbb{R}_{0,n}$  generated by  $(e_1, \dots, e_n)$ . A continuously differentiable function  $f$  in an open set  $\Omega$  of  $\mathbb{R}^{2n}$  with values in  $\mathbb{R}_{0,n}$  is called biregular in  $\Omega$  if and only if it satisfies in  $\Omega$  the system

$$\partial_{x_1}f = f\partial_{x_2} = 0.$$

The theory of biregular functions may be viewed as a natural generalization to higher dimension of the theory of holomorphic functions in  $\mathbb{C}^2$  (see [28, 29, 30, 95]).

The Liouville and Hartogs theorems for these functions (see [29]) enable us to state the analogue of Lemma 6.1.

**Lemma 6.4** *Suppose that  $K$  is a compact subset of  $\mathbb{R}^{2n}$  such that  $\mathbb{R}^{2n} \setminus K$  is connected. If  $f$  is biregular and bounded in  $\mathbb{R}^{2n} \setminus K$ , then  $f$  is a constant.*

*Proof.* The proof runs along similar lines as the proof of Lemma 6.1.  $\square$

Let  $\Sigma$  be an AD-regular surface and let  $\varphi$  be a regular majorant. Assume that  $f$  is an  $\mathbb{R}_{0,n}$ -valued function which belongs to  $H_\varphi(\Sigma)$ . It follows that

$$\mathbf{C}_\Sigma^{isot} f(\underline{x}) = \mathbf{M}_1 f(\underline{x}) + i\mathbf{M}_2 f(\underline{x}), \quad \underline{x} \in \mathbb{R}^{2n} \setminus \Sigma, \quad (6.8)$$

where

$$\mathbf{M}_1 f(\underline{x}) = \int_\Sigma E_1(\underline{y} - \underline{x}) \underline{\nu}_1(\underline{y}) f(\underline{y}) + f(\underline{y}) \underline{\nu}_2(\underline{y}) E_2(\underline{y} - \underline{x}) d\mathcal{H}^{2n-1}(\underline{y})$$

and

$$\mathbf{M}_2 f(\underline{x}) = \int_\Sigma E_1(\underline{y} - \underline{x}) \tilde{f}(\underline{y}) \underline{\nu}_2(\underline{y}) - \underline{\nu}_1(\underline{y}) \tilde{f}(\underline{y}) E_2(\underline{y} - \underline{x}) d\mathcal{H}^{2n-1}(\underline{y}).$$

It follows that  $\mathbf{M}_1$  and  $\mathbf{M}_2$  are harmonic in  $\mathbb{R}^{2n} \setminus \Sigma$  and that  $\mathbf{M}_1 f(\infty) = \mathbf{M}_2 f(\infty) = 0$ . Note that the integral operator  $\mathbf{M}_1$  plays the role of the Bochner-Martinelli integral in the theory of biregular functions.

We also have that

$$\mathbf{S}_\Sigma^{isot} f(\underline{z}) = \mathbf{N}_1 f(\underline{z}) + i\mathbf{N}_2 f(\underline{z}), \quad \underline{z} \in \Sigma, \quad (6.9)$$

where  $\mathbf{N}_1$  and  $\mathbf{N}_2$  are the singular versions of  $\mathbf{M}_1$  and  $\mathbf{M}_2$  respectively, given by

$$\mathbf{N}_1 f(\underline{z}) = 2 \lim_{\epsilon \rightarrow 0+} \mathbf{N}_{1,\epsilon} f(\underline{z}) + f(\underline{z}),$$

with

$$\begin{aligned} \mathbf{N}_{1,\epsilon} f(\underline{z}) = \int_{\Sigma \setminus B(\underline{z}, \epsilon)} & \left( E_1(\underline{y} - \underline{z}) \underline{\nu}_1(\underline{y}) (f(\underline{y}) - f(\underline{z})) \right. \\ & \left. + (f(\underline{y}) - f(\underline{z})) \underline{\nu}_2(\underline{y}) E_2(\underline{y} - \underline{z}) \right) d\mathcal{H}^{2n-1}(\underline{y}) \end{aligned}$$

and

$$\mathbf{N}_2 f(\underline{z}) = 2 \lim_{\epsilon \rightarrow 0+} \mathbf{N}_{2,\epsilon} f(\underline{z}),$$

with

$$\begin{aligned} \mathbf{N}_{2,\epsilon}f(\underline{z}) = \int_{\Sigma \setminus B(\underline{z},\epsilon)} & \left( E_1(\underline{y} - \underline{z})(\tilde{f}(\underline{y}) - \tilde{f}(\underline{z}))\nu_2(\underline{y}) \right. \\ & \left. - \nu_1(\underline{y})(\tilde{f}(\underline{y}) - \tilde{f}(\underline{z}))E_2(\underline{y} - \underline{z}) \right) d\mathcal{H}^{2n-1}(\underline{y}). \end{aligned}$$

On account of Theorem 5.3, we obtain:

**Theorem 6.6** *Let  $\Sigma$  be an AD-regular surface and let  $\varphi$  be a regular majorant. If  $f \in H_\varphi(\Sigma)$  is an  $\mathbb{R}_{0,n}$ -valued function, then  $\mathbf{M}_1f$  and  $\mathbf{M}_2f$  have continuous limit values on  $\Sigma$  given by*

$$\mathbf{M}_1^\pm f(\underline{z}) = \lim_{\Omega^\pm \ni \underline{x} \rightarrow \underline{z}} \mathbf{M}_1f(\underline{x}) = \frac{1}{2}(\mathbf{N}_1f(\underline{z}) \pm f(\underline{z})), \quad \underline{z} \in \Sigma, \quad (6.10)$$

$$\mathbf{M}_2^\pm f(\underline{z}) = \lim_{\Omega^\pm \ni \underline{x} \rightarrow \underline{z}} \mathbf{M}_2f(\underline{x}) = \frac{1}{2} \mathbf{N}_2f(\underline{z}), \quad \underline{z} \in \Sigma. \quad (6.11)$$

The following result can be considered as an analogue of Theorem 6.1.

**Theorem 6.7** *Assume that  $\Sigma$  is an AD-regular surface and let  $\varphi$  be a regular majorant. Then  $\mathbf{N}_1$  and  $\mathbf{N}_2$  are bounded operators mapping  $H_\varphi(\Sigma)$  into itself. Moreover, the formulae*

$$\begin{aligned} \mathbf{N}_1^2f - \mathbf{N}_2^2f &= f, \\ \mathbf{N}_1\mathbf{N}_2f + \mathbf{N}_2\mathbf{N}_1f &= 0, \end{aligned}$$

hold for all  $f \in H_\varphi(\Sigma)$ .

*Proof.* The proof easily follows using (6.9) as well as Theorems 5.6 and 5.7.

□

We now come to the main result of the section.

**Theorem 6.8** *Let  $\Sigma$  be an AD-regular surface and let  $\varphi$  be a regular majorant. Suppose that  $f \in H_\varphi(\Sigma)$  is an  $\mathbb{R}_{0,n}$ -valued function. The following statements are equivalent:*

- (i)  $f$  has a biregular extension to  $\Omega^+$ ;
- (ii)  $N_1 f = f$  on  $\Sigma$ ;
- (iii)  $M_1 f = 0$  in  $\Omega^-$ ;
- (iv)  $M_2 f = 0$  in  $\Omega^-$ ;
- (v)  $N_2 f = 0$  on  $\Sigma$ ;
- (vi)  $M_2 f = 0$  in  $\Omega^+$ .

*Proof.* (i)  $\Rightarrow$  (ii): Let  $F$  be a biregular extension of  $f$  to  $\Omega^+$ . By Corollary 5.2, we have that

$$F(\underline{x}) = M_1 f(\underline{x}), \quad \underline{x} \in \Omega^+.$$

From (6.10) it then follows that

$$f(\underline{z}) = \frac{1}{2}(N_1 f(\underline{z}) + f(\underline{z})), \quad \underline{z} \in \Sigma,$$

and hence  $N_1 f = f$  on  $\Sigma$ .

(ii)  $\Leftrightarrow$  (iii): If  $N_1 f = f$  on  $\Sigma$ , then from (6.10) we obtain  $M_1^- f = 0$  on  $\Sigma$ . Since  $M_1 f$  is harmonic in  $\Omega^-$  and vanishes at infinity, it thus follows that  $M_1 f = 0$  in  $\Omega^-$ . Using (6.10) we can easily deduce that (iii)  $\Rightarrow$  (ii).

(iii)  $\Leftrightarrow$  (iv): If  $M_1 f = 0$  in  $\Omega^-$ , then from (6.8) we see that  $M_2 f$  is isotonic in  $\Omega^-$  and hence biregular in  $\Omega^-$ . Therefore  $M_2 f = 0$  in  $\Omega^-$  by Lemma 6.4. In the same way we can prove (iv)  $\Rightarrow$  (iii).

(iv)  $\Leftrightarrow$  (v)  $\Leftrightarrow$  (vi): This follows from the maximum principle for harmonic functions and (6.11).

(vi)  $\Rightarrow$  (i): If  $M_2 f = 0$  in  $\Omega^+$ , then it follows from (6.8) that  $M_1 f$  is biregular in  $\Omega^+$ . Since (vi)  $\Rightarrow$  (ii), we can conclude that

$$F(\underline{x}) = \begin{cases} M_1 f(\underline{x}) & \text{for } \underline{x} \in \Omega^+, \\ f(\underline{x}) & \text{for } \underline{x} \in \Sigma, \end{cases}$$

is a biregular extension of  $f$  to  $\Omega^+$ . □



**Corollary 6.3** *Let  $\Sigma$  be an AD-regular surface and let  $\varphi$  be a regular majorant. Suppose that  $F$  is an  $\mathbb{R}_{0,n}$ -valued continuous function on  $\Omega^+$  such that  $f = F|_{\Sigma} \in H_{\varphi}(\Sigma)$ . A necessary and sufficient condition for  $F$  to be biregular in  $\Omega^+$  is that  $F = M_1 f$  in  $\Omega^+$ .*

*Proof.* The necessity follows from Corollary 5.2. We now proceed to prove the sufficiency. Suppose that  $F = M_1 f$  in  $\Omega^+$ , then from (6.10) we obtain  $N_1 f = f$  on  $\Sigma$ , whence  $M_2 f = 0$  in  $\Omega^+$  on account of Theorem 6.8. Hence  $M_1 f$  is biregular in  $\Omega^+$ .  $\square$

## 6.5 Biregular extension for continuous functions

In this final section, we will concern the case of  $\mathbb{R}_{0,n}$ -valued continuous functions on  $\Sigma$ .

The proofs of the following results are very similar to those given in Section 6.3 and will be omitted.

**Theorem 6.9** *Let  $\Sigma$  be an AD-regular surface and let  $f$  be an  $\mathbb{R}_{0,n}$ -valued continuous function on  $\Sigma$ . If the integrals*

$$\int_{\Sigma \cap B(\underline{z}, \epsilon)} \left( E_1(\underline{y} - \underline{z}) \nu_1(\underline{y}) (f(\underline{y}) - f(\underline{z})) \right. \\ \left. + (f(\underline{y}) - f(\underline{z})) \nu_2(\underline{y}) E_2(\underline{y} - \underline{z}) \right) d\mathcal{H}^{2n-1}(\underline{y})$$

*converge uniformly to zero for  $\underline{z} \in \Sigma$  as  $\epsilon \rightarrow 0$ , then  $M_1 f$  has continuous limit values on  $\Sigma$  given by (6.10).*

**Theorem 6.10** *Let  $\Sigma$  be an AD-regular surface and let  $f$  be an  $\mathbb{R}_{0,n}$ -valued continuous function on  $\Sigma$ . If the integrals*

$$\int_{\Sigma \cap B(\underline{z}, \epsilon)} \left( E_1(\underline{y} - \underline{z}) (f(\underline{y}) - f(\underline{z})) \nu_2(\underline{y}) \right. \\ \left. - \nu_1(\underline{y}) (f(\underline{y}) - f(\underline{z})) E_2(\underline{y} - \underline{z}) \right) d\mathcal{H}^{2n-1}(\underline{y}).$$

converge uniformly to zero for  $\underline{z} \in \Sigma$  as  $\epsilon \rightarrow 0$ , then  $M_2 f$  has continuous limit values on  $\Sigma$  given by (6.11).

**Lemma 6.5** *Let  $f$  be an  $\mathbb{R}_{0,n}$ -valued continuous function on  $\Sigma$ ,  $\underline{z} \in \Sigma$  and  $\epsilon > 0$ .*

(i) *If  $\underline{x} \in \Omega^+$  is such that  $|\underline{x} - \underline{z}| = \epsilon/2$ , then we have that*

$$\begin{aligned} & |M_1 f(\underline{x}) - N_{1,\epsilon} f(\underline{z}) - f(\underline{z})| \\ & \leq C \left( \frac{\theta_{\underline{z}}(\epsilon)}{(\text{dist}(\underline{x}, \Sigma))^{2n-1}} \omega_f(\epsilon) + \epsilon \int_{\epsilon}^d \frac{\omega_f(\tau)}{\tau^{2n}} d\theta_{\underline{z}}(\tau) \right). \end{aligned}$$

(ii) *If  $\underline{x} \in \Omega^-$  is such that  $|\underline{x} - \underline{z}| = \epsilon/2$ , then we have that*

$$\begin{aligned} & |M_1 f(\underline{x}) - N_{1,\epsilon} f(\underline{z})| \\ & \leq C \left( \frac{\theta_{\underline{z}}(\epsilon)}{(\text{dist}(\underline{x}, \Sigma))^{2n-1}} \omega_f(\epsilon) + \epsilon \int_{\epsilon}^d \frac{\omega_f(\tau)}{\tau^{2n}} d\theta_{\underline{z}}(\tau) \right). \end{aligned}$$

**Lemma 6.6** *Let  $f$  be an  $\mathbb{R}_{0,n}$ -valued continuous function on  $\Sigma$ ,  $\underline{z} \in \Sigma$  and  $\epsilon > 0$ . Then we have that*

$$|M_2 f(\underline{x}) - N_{2,\epsilon} f(\underline{z})| \leq C \left( \frac{\theta_{\underline{z}}(\epsilon)}{(\text{dist}(\underline{x}, \Sigma))^{2n-1}} \omega_f(\epsilon) + \epsilon \int_{\epsilon}^d \frac{\omega_f(\tau)}{\tau^{2n}} d\theta_{\underline{z}}(\tau) \right),$$

where  $\underline{x} \in \Omega^{\pm}$  is such that  $|\underline{x} - \underline{z}| = \epsilon/2$ .

We can now state the main results of this section which may be considered as generalizations of the results obtained by Kytmanov and Aizenberg (see [76]) to the case of biregular functions.

**Theorem 6.11** *Let  $\Omega^+$  be a Lipschitz domain. Suppose that  $f$  is an  $\mathbb{R}_{0,n}$ -valued continuous function on  $\Sigma$ . Then  $f$  has a biregular extension to  $\Omega^+$  if and only if  $M_1 f = 0$  in  $\Omega^-$ .*

*Proof.* The necessity is obvious, so we prove the sufficiency. Suppose that  $M_1f = 0$  in  $\Omega^-$ . It follows that  $M_2f$  is biregular in  $\Omega^-$ , whence  $M_2f = 0$  in  $\Omega^-$  on account of Lemma 6.4.

Now if  $\underline{z} \in \Sigma$  and  $\epsilon > 0$ , then by Lemmas 6.5 and 6.6 we get that

$$|N_{1,\epsilon}f(\underline{z})| \leq C \left( \frac{\theta_{\underline{z}}(\epsilon)}{(\text{dist}(\underline{x}, \Sigma))^{2n-1}} \omega_f(\epsilon) + \epsilon \int_{\epsilon}^d \frac{\omega_f(\tau)}{\tau^{2n}} d\theta_{\underline{z}}(\tau) \right),$$

$$|N_{2,\epsilon}f(\underline{z})| \leq C \left( \frac{\theta_{\underline{z}}(\epsilon)}{(\text{dist}(\underline{x}, \Sigma))^{2n-1}} \omega_f(\epsilon) + \epsilon \int_{\epsilon}^d \frac{\omega_f(\tau)}{\tau^{2n}} d\theta_{\underline{z}}(\tau) \right),$$

where  $\underline{x} \in \Omega^-$  is such that  $|\underline{x} - \underline{z}| = \epsilon/2$ . When  $\underline{x}$  approaches  $\underline{z}$  non-tangentially inside the cone  $V(\underline{z})$ , the uniform interior and exterior cone condition implies that

$$\frac{\epsilon}{2} = |\underline{x} - \underline{z}| \leq (1 + \lambda) \text{dist}(\underline{x}, \Sigma).$$

Combining the above inequality with the AD-regularity of  $\Sigma$ , we obtain

$$|N_{1,\epsilon}f(\underline{z})| \leq C \left( \omega_f(\epsilon) + \epsilon \int_{\epsilon}^d \frac{\omega_f(\tau)}{\tau^2} d\tau \right),$$

$$|N_{2,\epsilon}f(\underline{z})| \leq C \left( \omega_f(\epsilon) + \epsilon \int_{\epsilon}^d \frac{\omega_f(\tau)}{\tau^2} d\tau \right).$$

Therefore  $N_{1,\epsilon}f(\underline{z})$  and  $N_{2,\epsilon}f(\underline{z})$  converge uniformly on  $\Sigma$  as  $\epsilon \rightarrow 0$ . Theorems 6.9 and 6.10 now imply that  $M_1f$  and  $M_2f$  have continuous limit values on  $\Sigma$  given by (6.10) and (6.11) respectively. It then follows that  $M_1f$  is biregular in  $\Omega^+$  and  $N_1f = f$  on  $\Sigma$ .  $\square$

In the same spirit we can also prove:

**Theorem 6.12** *Let  $\Omega^+$  be a Lipschitz domain. Suppose that  $F$  is an  $\mathbb{R}_{0,n}$ -valued continuous function on  $\overline{\Omega^+}$ . A necessary and sufficient condition for  $F$  to be biregular in  $\Omega^+$  is that  $F = M_1F$  in  $\Omega^+$ .*



# Conclusion

This thesis contains the following contributions to the further development of Euclidean and Hermitean Clifford analysis:

- A new class of monogenic functions called steering monogenic functions.
- CK-extensions around special surfaces of codimension 2.
- An alternative proof for and a new generalization of Fueter's theorem for monogenic functions.
- A closed formula for the CK-extension of the Gauss-distribution in  $\mathbb{R}^m$ .
- A differential and integral criterion for the existence of a Hermitean monogenic extension of a continuous function on a surface in  $\mathbb{R}^m$ ,  $m = 2n$ .
- Formulae for the square of the Bochner-Martinelli singular integral operator and for its higher dimensional version in the theory of biregular functions.
- The Plemelj-Sokhotski formulae for the Bochner-Martinelli integral and for its higher dimensional version in the theory of biregular functions.

- An alternative holomorphic extension theorem for continuous functions on non-smooth surfaces.
- The generalization of Aronov-Kytmanov-Aĭzenberg type theorems to the case of biregular functions.

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